

# Stochastic Constrained Navier-Stokes Equations on $\mathbb{T}^2$

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## Abstract

We study constrained 2-dimensional Navier-Stokes Equations driven by a multiplicative Gaussian noise in the Stratonovich form. In the deterministic case [1] we showed the existence of global solutions only on a two dimensional torus and hence we concentrated on such a case here. We prove the existence of a martingale solution and later using Schmalz idea [16] we show the pathwise uniqueness of the solutions. We also establish the existence of a strong solution using some results from Ondreját [13].

*Keywords:* Stochastic Navier-Stokes, constrained energy, periodic boundary conditions, martingale solution, strong solution.

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## 1. Introduction

In the present article we consider the stochastic Navier-Stokes equations

$$(1.1) \quad \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = \sum_{j=1}^m (c_j(x) \cdot \nabla)u \circ dW_j(t), \quad t \in [0, \infty)$$

in  $\mathcal{O} = [0, 2\pi]^2$  with periodic boundary conditions and with the incompressibility condition

$$\operatorname{div} u = 0.$$

This problem can be identified as a problem on a two-dimensional torus  $\mathbb{T}^2$  what we will assume to be our case. Here  $u : [0, \infty) \times \mathcal{O} \rightarrow \mathbb{R}^2$  and  $p : [0, \infty) \times \mathcal{O} \rightarrow \mathbb{R}$  represent the

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velocity and the pressure of the fluid. Furthermore  $\sum_{j=1}^m (c_j(x) \cdot \nabla)u \circ dW_j(t)$  stands for the random forcing, where  $c_j$ ,  $j = 1, \dots, m$ , are divergence free vector fields (so that the corresponding transport operators  $\tilde{C}_j u := (c_j(x) \cdot \nabla)u$  are skew symmetric in  $L^2(\mathbb{T}^2, \mathbb{R}^2)$ ) and  $W_j$ ,  $j = 1, \dots, m$  are independent  $\mathbb{R}$ -valued standard Brownian Motions.

The above problem projected on  $H \cap \mathcal{M}$  can be written in an abstract form as the following initial value problem

$$(1.2) \quad \begin{cases} du(t) + \nu A u(t) dt + B(u(t)) dt = \nu |\nabla u(t)|_{L^2}^2 u(t) dt + \sum_{j=1}^m C_j u(t) \circ dW_j(t), & t \in [0, T], \\ u(0) = u_0, \end{cases}$$

where  $H$  is the space of square integrable, divergence free and mean zero vector fields on  $\mathcal{O}$  and

$$\mathcal{M} = \{u \in H : |u|_{L^2} = 1\}.$$

Here  $A$  and  $B$  are appropriate maps corresponding to the Laplacian and the nonlinear term, respectively in the Navier-Stokes equations, see Section 2 and  $C_j = \Pi(\tilde{C}_j)$ , where  $\Pi : L^2(\mathbb{T}^2, \mathbb{R}) \rightarrow H$  is the Leray-Helmholtz projection operator [17] that projects the square integrable vector fields onto the divergence free vector field.

We prove the existence and uniqueness of a strong solution. The construction of a solution is based on the classical Faedo-Galerkin approximation, i.e.

$$(1.3) \quad \begin{cases} du_n(t) = - [P_n A u_n(t) + B_n(u_n(t)) - |\nabla u_n(t)|_{L^2}^2 u_n(t)] dt \\ \quad + \sum_{j=1}^m P_n C_j u_n(t) \circ dW_j(t), & t \in [0, T], \\ u_n(0) = \frac{P_n u_0}{|P_n u_0|} \end{cases}$$

given in Section 5. Let us point out that without the normalisation of the initial condition in the above problem (1.3), the solution may not be a global one. The crucial point is to prove suitable uniform a priori estimates on the sequence  $u_n$ . We will prove that the following estimates hold

$$\sup_{n \geq 1} \mathbb{E} \left[ \int_0^T |u_n(s)|_{D(A)}^2 ds \right] < \infty,$$

and

$$\sup_{n \geq 1} \mathbb{E} \left( \sup_{0 \leq s \leq T} \|u_n(s)\|_V^{2p} \right) < \infty,$$

for  $p \in [1, 1 + \frac{1}{K_c})$ , where  $D(A)$  is the domain of the Stokes operator and  $V = D((A^{1/2}))$ , see Section 2 for precise definitions and the positive constant  $K_c$  is defined in Lemma 5.4.

In Theorem 3.4 we prove the existence of a martingale solution using the tightness criterion in the topological space  $\mathcal{Z}_T = \mathcal{C}([0, T]; H) \cap L^2_w(0, T; D(A)) \cap L^2(0, T; V) \cap \mathcal{C}([0, T]; V_w)$  showing that the trajectories of the solution lie in  $\mathcal{C}([0, T]; V_w)$  but later on in Lemma 3.6 we show that in fact the trajectories lie in  $\mathcal{C}([0, T]; V)$ .

Our work is an extension of a recent preprint by the two authors and Mauro Mariani [1] from the deterministic to a stochastic setting. More information and motivation can also be found therein. Let us recall that already in the deterministic setting, we have been able to prove the global existence of solutions only for CNSEs only with periodic boundary conditions and this is why we have concentrated here on such a case. A similar problem for stochastic heat equation with polynomial drift but with a different type of noise has recently been a subject of a PhD thesis by Javed Hussain [7]. It's remarkable that in that case the result holds for Dirichlet boundary conditions as well.

We consider the noise of gradient type in the Stratonovich form (1.1). The structure of noise is such that it is tangent to the manifold  $\mathcal{M}$  just like the non-linear part from Navier-Stokes and hence there is no contribution to the equation (1.2) because of the constraint. In the deterministic setting [1] we proved the existence of a global solution by proving the existence of a local solution using Banach Fixed Point Theorem; and no explosion principle, i.e enstrophy ( $V$ - norm) of the solution remains bounded. We can't take the similar approach in the stochastic setting as one can't prove the existence of a local solution using the Banach Fixed Point Theorem and hence we switch to more classical approach of proving the existence of a solution using the Faedo-Galerkin approximation.

We consider the Faedo-Galerkin approximation (5.2) of (1.2). We prove that each approximated equation has a global solution. One can show that for every  $n \in \mathbb{N}$  global solution to (5.2) exist for all domains, in particular for Dirichlet boundary conditions. But in order to obtain a priori estimates, Lemma 5.4 we need to consider the Navier-Stokes Equations (NSEs) on a two dimensional torus  $\mathbb{T}^2$  (i.e. the NSEs with the periodic boundary conditions).

In order to prove that the laws of the solution of these approximated equations are tight on  $\mathcal{Z}_T$  (defined in (4.1)), apart from a priori estimates we also need the Aldous condition, Definition 4.4. After proving that the laws are tight in Lemma 5.5, by the application of Prokhorov, Jakubowski-Skorokhod Theorem and the Martingale Representation Theorem we prove Theorem 3.4. The paper is organised in the following way:

In Section 2 we introduce some functional spaces and certain linear operators along with the well-established estimates. Stochastic Constrained Navier-Stokes Equations (SCNSEs) are introduced in Section 3 along with the definitions of a martingale solution and strong solution and all the important results of this paper. Section 4 contains all the well-known and already established results regarding compactness. In Section 5 we establish certain

estimates on the way to prove Theorem 3.4. We conclude the paper by proving the existence and uniqueness of a strong solution using the results from Ondrejat [13] in Section 6.

## 2. Functional setting

Let  $\mathcal{O} \subset \mathbb{R}^2$  be a bounded domain with periodic boundary conditions. Let  $p \in [1, \infty)$  and let  $\mathbf{L}^p(\mathcal{O}) = L^p(\mathcal{O}, \mathbb{R}^2)$  denote the Banach space of Lebesgue measurable  $\mathbb{R}^2$ -valued  $p$ -th power integrable functions on the set  $\mathcal{O}$ . The norm in  $\mathbf{L}^p(\mathcal{O})$  is given by

$$|u|_{L^p} := \left( \int_{\mathcal{O}} |u(x)|^p dx \right)^{\frac{1}{p}}, \quad u \in \mathbf{L}^p(\mathcal{O}).$$

By  $\mathbf{L}^\infty(\mathcal{O}) = L^\infty(\mathcal{O}, \mathbb{R}^2)$  we denote the Banach space of lebesgue measurable essentially bounded  $\mathbb{R}^2$ -valued functions defined on  $\mathcal{O}$ . The norm is given by

$$|u|_{\mathbf{L}^\infty(\mathcal{O})} := \text{esssup} \{|u(x)|, x \in \mathcal{O}\}, \quad u \in \mathbf{L}^\infty(\mathcal{O}).$$

If  $p = 2$ , then  $\mathbf{L}^2(\mathcal{O}) = L^2(\mathcal{O}, \mathbb{R}^2)$  is a Hilbert space with the scalar product given by

$$\langle u, v \rangle_{L^2} := \int_{\mathcal{O}} u(x) \cdot v(x) dx, \quad u, v \in \mathbf{L}^2(\mathcal{O}).$$

Let  $k \in \mathbb{N}$ ,  $p \in [1, \infty)$ . By  $\mathbf{W}^{k,p}(\mathcal{O}) = W^{k,p}(\mathcal{O}, \mathbb{R}^2)$  we denote the Sobolev space of all  $u \in \mathbf{L}^p(\mathcal{O})$  for which there exist weak derivatives  $D^\alpha u \in \mathbf{L}^p(\mathcal{O})$ ,  $|\alpha| \leq k$ . For  $p = 2$ , we will write  $W^{k,2}(\mathcal{O}, \mathbb{R}^2) =: H^k$  and will denote it's norm by  $\|\cdot\|_{H^k}$ . In particular  $H^1$  is a Hilbert space with the scalar product given by

$$\langle u, v \rangle_{H^1} := \langle u, v \rangle_{L^2} + \langle \nabla u, \nabla v \rangle_{L^2}, \quad u, v \in H^1(\mathcal{O}).$$

Let  $\mathcal{C}_c^\infty(\mathcal{O}, \mathbb{R}^2)$  denote the space of all  $\mathbb{R}^2$ -valued functions of class  $\mathcal{C}^\infty$  with compact supports contained in  $\mathcal{O}$ . We introduce the following spaces:

$$\begin{aligned} \mathcal{V} &= \{u \in \mathcal{C}_c^\infty(\mathcal{O}, \mathbb{R}^2) : \nabla \cdot u = 0\}, \\ \mathbb{L}_0^2 &= \left\{ u \in L^2(\mathbb{T}^2, \mathbb{R}^2) : \int_{\mathbb{T}^2} u(x) dx = 0 \right\}, \\ \mathbb{H} &= \{u \in \mathbb{L}_0^2 : \nabla \cdot u = 0\}, \\ \mathbb{V} &= H^1 \cap \mathbb{H}. \end{aligned} \tag{2.1}$$

We endow  $\mathbb{H}$  with the scalar product and norm of  $L^2$  and denote it by

$$\langle u, v \rangle_{\mathbb{H}} := \langle u, v \rangle_{L^2}, \quad |u|_{\mathbb{H}} := |u|_{L^2}, \quad u, v \in \mathbb{H}.$$

We equip the space  $V$  with the scalar product  $\langle u, v \rangle_V := \langle \nabla u, \nabla v \rangle_H$  and norm  $\|u\|_V, u, v \in V$ .

One can show that in the case of  $\mathcal{O} = \mathbb{T}^2$ ,  $V$ -norm  $\|\cdot\|_V$ , and  $H^1$ -norm  $\|\cdot\|_{H^1}$  are equivalent on  $V$ .

We denote by  $A : D(A) \rightarrow H$ , the Stokes operator which is defined by

$$\begin{aligned} D(A) &= H \cap H^2(\mathbb{T}^2), \\ Au &= -\Delta u, \quad u \in D(A). \end{aligned}$$

$D(A)$  is a Hilbert space under the graph norm,

$$|u|_{D(A)}^2 := |u|_H^2 + |Au|_{L^2}^2.$$

It is well known that  $A$  is a self adjoint positive operator in  $H$ . Moreover

$$D(A^{1/2}) = V \quad \text{and} \quad \langle Au, u \rangle_H = \|u\|_V^2 = |\nabla u|_{L^2}^2, \quad u \in D(A).$$

We introduce a continuous tri-linear form  $b : L^p \times W^{1,q} \times L^r \rightarrow \mathbb{R}$ ,

$$b(u, v, w) = \sum_{i,j=1}^2 \int_{\mathcal{O}} u^i \frac{\partial v^j}{\partial x^i} w^j \, dx, \quad u \in L^p, v \in W^{1,q}, w \in L^r$$

where  $p, q, r \in [1, \infty]$  satisfies

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1.$$

By the Sobolev Embedding Theorem and the Hölder inequality, we obtain the following estimates

$$\begin{aligned} (2.2) \quad |b(u, v, w)| &\leq |u|_{L^4} \|v\|_V |w|_{L^4}, \quad u, w \in L^4, v \in V, \\ &\leq c \|u\|_V \|v\|_V \|w\|_V, \quad u, v, w \in V. \end{aligned}$$

We can define a bilinear map  $B : V \times V \rightarrow V'$  such that

$$\langle B(u, v), \phi \rangle = b(u, v, \phi), \quad \text{for } u, v, \phi \in V,$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality between  $V$  and  $V'$ . The following inequality is well known [17]:

$$(2.3) \quad |b(u, v, \phi)| \leq \sqrt{2} |u|_H^{\frac{1}{2}} \|u\|_V^{\frac{1}{2}} \|v\|_V^{\frac{1}{2}} |v|_{D(A)}^{\frac{1}{2}} |\phi|_H, \quad u \in V, v \in D(A), \phi \in H.$$

Thus  $b$  can be uniquely extended to the tri-linear form (denoted by the same letter)

$$b : V \times D(A) \times H \rightarrow \mathbb{R}.$$

We can now also extend the operator  $B$  uniquely to a bounded bilinear operator

$$B : V \times D(A) \rightarrow H.$$

The following properties of the tri-linear map  $b$  and the bilinear map  $B$  are very well established in [1, 17],

$$(2.4) \quad \begin{aligned} b(u, u, u) &= 0, & u &\in V, \\ b(u, w, w) &= 0, & u &\in V, w \in H^1, \\ \langle B(u, u), Au \rangle_H &= 0, & u &\in D(A). \end{aligned}$$

We will also use the following notation,  $B(u) := B(u, u)$ .

The 2D Navier-Stokes equations driven by multiplicative Gaussian noise in the Stratonovich form are given as following:

$$(2.5) \quad \begin{cases} \frac{\partial u(x, t)}{\partial t} - \nu \Delta u(x, t) + (u(x, t) \cdot \nabla) u(x, t) + \nabla p(x, t) \\ \quad = \sum_{j=1}^m [(c_j(x) \cdot \nabla) u(x, t)] \circ dW_j(t), & t > 0, x \in \mathcal{O}, \\ \operatorname{div} u(\cdot, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), & x \in \mathcal{O}, \end{cases}$$

$u : [[0, \infty) \times \mathcal{O} \rightarrow \mathbb{R}^2$  and  $p : [[0, \infty) \times \mathcal{O} \rightarrow \mathbb{R}$  are velocity and pressure of the fluid respectively.  $\nu$  is the viscosity of the fluid (with no loss of generality,  $\nu$  will be taken equal to 1 for the rest of the article). Here we assume that  $c_j : \mathcal{O} \rightarrow \mathbb{R}^2$  are sufficiently smooth divergence free vector fields,  $W_j$  are  $\mathbb{R}$ -valued i.i.d. standard Brownian motions and  $\circ$  denotes the Stratonovich form. Note that the operators  $\tilde{C}_j$ ,  $j \in \{1, \dots, m\}$ , defined by  $\tilde{C}_j u := (c_j(x) \cdot \nabla) u$ , for  $u \in V$  are skew-symmetric on  $L^2(\mathbb{T}^2, \mathbb{R}^2)$ , i.e.  $\tilde{C}_j^* = -\tilde{C}_j$ , where  $\tilde{C}_j^*$  denotes the adjoint of  $\tilde{C}_j$  on  $L^2(\mathbb{T}^2, \mathbb{R}^2)$ .

We will be frequently using the following short-cut notation

$$Cu \circ dW(t) = \sum_{j=1}^m C_j u(t) \circ dW_j(t),$$

where  $C_j = \Pi(\tilde{C}_j)$  and  $\Pi$  is the Leray-Helmholtz projection operator.

With all the notations as defined above, the Navier-Stokes equation (2.5) projected on divergence free vector field is given by

$$(2.6) \quad \begin{cases} du(t) + [Au(t) + B(u(t))] dt = Cu(t) \circ dW(t), \\ u(0) = u_0. \end{cases}$$

Let us denote the set of divergence free  $\mathbb{R}^2$ -valued functions with unit  $L^2$  norm, as following

$$\mathcal{M} = \{u \in H : |u|_{L^2} = 1\}.$$

Then the tangent space at  $u$  is defined as,

$$T_u \mathcal{M} = \{v \in H : \langle v, u \rangle_H = 0\}, \quad u \in \mathcal{M}.$$

We define a linear map  $\pi_u : H \rightarrow T_u \mathcal{M}$  by

$$\pi_u(v) = v - \langle v, u \rangle_H u,$$

then  $\pi_u$  is the orthogonal projection from  $H$  into  $T_u \mathcal{M}$ .

Since for every  $j \in \{1, \dots, m\}$ ,  $C_j^* = -C_j$  in  $H$  we infer that

$$(2.7) \quad \langle C_j u, u \rangle_H = 0, \quad u \in V, \quad j \in \{1, \dots, m\}.$$

In particular, if  $u \in V \cap \mathcal{M}$ , then  $C_j u \in T_u \mathcal{M}$  for every  $j \in \{1, \dots, m\}$  and hence won't produce any correction terms when projected on the tangent space  $T_u \mathcal{M}$ , which is shown explicitly below.

Let  $F(u) = Au + B(u, u) - Cu \circ dW(t)$  and  $\hat{F}(u)$  be the projection of  $F(u)$  on the tangent space  $T_u \mathcal{M}$ , then

$$\begin{aligned} \hat{F}(u) &= \pi_u(F(u)) \\ &= F(u) - \langle F(u), u \rangle_H u \\ &= Au + B(u) - Cu \circ dW - \langle Au + B(u) - Cu \circ dW, u \rangle_H u \\ &= Au - \langle Au, u \rangle_H u + B(u) - \langle B(u), u \rangle_H u - Cu \circ dW + \langle Cu, u \rangle_H \circ dW u \\ &= Au - |\nabla u|_{L^2}^2 u + B(u) - Cu \circ dW. \end{aligned}$$

The last equality follows from (2.7) and the identity that  $\langle B(u), u \rangle_H = 0$ .

**Remark 2.1.** Since  $\langle B(u), u \rangle_H = 0$  and  $u \in \mathcal{M}$ ,  $B(u) \in T_u \mathcal{M}$ .

Thus by projecting NSEs (2.6) on the manifold  $\mathcal{M}$ , we obtain the following Stochastic Constrained Navier-Stokes Equations (SCNSEs)

$$(2.8) \quad \begin{cases} du(t) + [Au(t) + B(u(t))] dt = |\nabla u(t)|_{L^2}^2 u(t) dt + Cu(t) \circ dW(t), \\ u(0) = u_0 \in V \cap \mathcal{M}. \end{cases}$$

### 3. Stochastic Constrained Navier-Stokes equations

We consider the following stochastic evolution equation

$$(3.1) \quad \begin{cases} du(t) + [Au(t) + B(u(t))] dt = |\nabla u(t)|_{L^2}^2 u(t) dt + Cu(t) \circ dW(t), & t \in [0, T], \\ u(0) = u_0, \end{cases}$$

where  $Cu(t, x) \circ dW(t) := \sum_{j=1}^m C_j u(t, x) \circ dW_j(t)$  with  $C_j u = \Pi((c_j(x) \cdot \nabla)u)$  and  $W_j, j = 1, \dots, m$ , are i.i.d standard  $\mathbb{R}$ -valued Brownian Motions.

**Assumptions.** We assume that

(A.1) Vector fields  $c_j : \mathcal{O} \rightarrow \mathbb{R}^2$  belong to the domain  $D(A)$  and  $Ac_j$  are bounded. In particular,  $c_j$  are bounded and divergence free vector fields.

(A.2)  $u_0 \in V \cap \mathcal{M}$ .

**Definition 3.1.** A stochastic basis  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  is a probability space equipped with the filtration  $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$  of it's  $\sigma$ -field  $\mathcal{F}$ .

**Definition 3.2.** We say that problem (3.1) has a **strong solution** iff for every stochastic basis  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  and every  $\mathbb{R}^m$ -valued  $\mathbb{F}$ -Wiener process  $W = (W(t))_{t \geq 0}$ , there exists a  $\mathbb{F}$ -progressively measurable process  $u : [0, T] \times \Omega \rightarrow D(A)$  with  $\mathbb{P}$ -a.e. paths

$$u(\cdot, \omega) \in \mathcal{C}([0, T]; V) \cap L^2(0, T; D(A)),$$

such that for all  $t \in [0, T]$  and all  $v \in V$   $\hat{\mathbb{P}}$ -a.s.

$$(3.2) \quad \begin{aligned} & \langle u(t), v \rangle - \langle u_0, v \rangle + \int_0^t \langle Au(s), v \rangle ds + \int_0^t \langle B(u(s)), v \rangle ds \\ &= \int_0^t |\nabla u(s)|_{L^2}^2 \langle u(s), v \rangle ds + \frac{1}{2} \int_0^t \sum_{j=1}^m \langle C_j^2 u(s), v \rangle ds + \int_0^t \sum_{j=1}^m \langle C_j u(s), v \rangle d\hat{W}_j(s). \end{aligned}$$



**Definition 3.3.** We say that there exists a **martingale solution** of (3.1) iff there exist

- a stochastic basis  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ ,
- an  $\mathbb{R}^m$ -valued  $\hat{\mathbb{F}}$ -Wiener process  $\hat{W}$ ,
- and a  $\hat{\mathbb{F}}$ -progressively measurable process  $u : [0, T] \times \hat{\Omega} \rightarrow D(A)$  with  $\hat{\mathbb{P}}$ -a.e. paths

$$u(\cdot, \omega) \in \mathcal{C}([0, T]; V_w) \cap L^2(0, T; D(A)),$$

such that for all  $t \in [0, T]$  and all  $v \in V$  the identity (3.2) holds  $\mathbb{P}$ -a.s.

Next we state some important results of this paper which will be proved in further sections.

**Theorem 3.4.** *Let assumptions (A.1) – (A.2) be satisfied. Then there exists a martingale solution  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, u)$  of problem (3.1) such that*

$$(3.3) \quad \hat{\mathbb{E}} \left[ \sup_{t \in [0, T]} \|u(t)\|_V^2 + \int_0^T |u(t)|_{D(A)}^2 dt \right] < \infty.$$

**Remark 3.5.** The solution obtained in the above theorem is weak in probabilistic sense and strong in PDE sense.

The next lemma shows that almost all the trajectories of the solution obtained in Theorem 3.4 are almost everywhere equal to a continuous  $V$ -valued function defined on  $[0, T]$ .

**Lemma 3.6.** *Assume that the assumptions (A.1)–(A.2) are satisfied. Let  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, \hat{W}, u)$  be a martingale solution of (3.1) such that*

$$(3.4) \quad \hat{\mathbb{E}} \left[ \sup_{t \in [0, T]} \|u(t)\|_V^2 + \int_0^T |u(s)|_{D(A)}^2 ds \right] < \infty.$$

*Then for  $\hat{\mathbb{P}}$  almost all  $\omega \in \hat{\Omega}$  the trajectory  $u(\cdot, \omega)$  is almost everywhere equal to a continuous  $V$ -valued function defined on  $[0, T]$ . Moreover for every  $t \in [0, T]$ ,  $\hat{\mathbb{P}}$ -a.s.*

$$(3.5) \quad \begin{aligned} u(t) = & u_0 - \int_0^t [Au(s) + B(u(s)) - |\nabla u(s)|_{L^2}^2 u(s)] ds \\ & + \frac{1}{2} \int_0^t \sum_{j=1}^m C_j^2 u(s) ds + \int_0^t \sum_{j=1}^m C_j u(s) d\hat{W}(s). \end{aligned}$$

**Definition 3.7.** Let  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, u^i), i = 1, 2$  be the martingale solutions of (3.1) with  $u^i(0) = u_0, i = 1, 2$ . Then we say that the solutions are **pathwise unique** if  $\mathbb{P}$ -a.s. for all  $t \in [0, T], u^1(t) = u^2(t)$ .

In Lemma 6.1 we will show that the pathwise uniqueness property for our problem holds. This will enable us to deduce the following theorem that summarises the main result of our paper:

**Theorem 3.8.** *For every  $u_0 \in V$  there exists a pathwise unique strong solution  $u$  of stochastic constrained Navier-Stokes equation (3.1) such that*

$$(3.6) \quad \mathbb{E} \left[ \int_0^T |u(t)|_{D(A)}^2 dt + \sup_{t \in [0, T]} \|u(t)\|_V^2 \right] < \infty.$$

**Remark 3.9.** The solution of (3.1) obtained in previous theorem is strong in both probabilistic and PDE sense.

#### 4. Compactness

Let us consider the following functional spaces:

$\mathcal{C}([0, T]; H) :=$  the space of continuous functions  $u : [0, T] \rightarrow H$  with the topology  $\mathcal{T}_1$  induced by the norm  $|u|_{\mathcal{C}([0, T]; H)} := \sup_{t \in [0, T]} |u(t)|_H$ ,

$L_w^2(0, T; D(A)) :=$  the space  $L^2(0, T; D(A))$  with the weak topology  $\mathcal{T}_2$ ,

$L^2(0, T; V) :=$  the space of measurable functions  $u : [0, T] \rightarrow V$  such that

$$|u|_{L^2(0, T; V)} = \left( \int_0^T \int_{\mathcal{O}} |\nabla u(x)|^2 dx dt \right)^{\frac{1}{2}} < \infty,$$

with the topology  $\mathcal{T}_3$  induced by the norm  $|u|_{L^2(0, T; V)}$ ,

$\mathcal{C}([0, T]; V_w) :=$  the space of weakly continuous functions  $u : [0, T] \rightarrow V$  endowed with the weakest topology  $\mathcal{T}_4$  such that for all  $h \in V$  the mappings

$$\mathcal{C}([0, T]; V_w) \ni u \rightarrow \langle u(\cdot), h \rangle_V \in \mathcal{C}([0, T]; \mathbb{R})$$

are continuous.

Let

$$(4.1) \quad \mathcal{Z}_T = \mathcal{C}([0, T]; H) \cap L_w^2(0, T; D(A)) \cap L^2(0, T; V) \cap \mathcal{C}(0, T; V_w),$$

and let  $\mathcal{T}$  be the supremum of the corresponding topologies.

#### 4.1. Tightness

Let  $(\mathbb{S}, \varrho)$  be a separable and complete metric space.

**Definition 4.1.** Let  $u \in \mathcal{C}([0, T]; \mathbb{S})$ . The modulus of continuity of  $u$  on  $[0, T]$  is defined by

$$m(u, \delta) := \sup_{s, t \in [0, T], |t-s| \leq \delta} \varrho(u(t), u(s)), \quad \delta > 0.$$

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space with filtration  $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$  satisfying the usual conditions, see [11], and let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of continuous  $\mathbb{F}$ -adapted  $\mathbb{S}$ -valued processes.

**Definition 4.2.** We say that the sequence  $(X_n)_{n \in \mathbb{N}}$  of  $\mathbb{S}$ -valued random variables satisfies condition **[T]** iff  $\forall \varepsilon > 0, \forall \eta > 0, \exists \delta > 0$ :

$$(4.2) \quad \sup_{n \in \mathbb{N}} \mathbb{P} \{m(X_n, \delta) > \eta\} \leq \varepsilon.$$

**Lemma 4.3.** Assume that  $(X_n)_{n \in \mathbb{N}}$  satisfies condition **[T]**. Let  $\mathbb{P}_n$  be the law of  $X_n$  on  $\mathcal{C}([0, T]; \mathbb{S}), n \in \mathbb{N}$ . Then for every  $\varepsilon > 0$  there exists a subset  $A_\varepsilon \subset \mathcal{C}([0, T]; \mathbb{S})$  such that

$$\sup_{n \in \mathbb{N}} \mathbb{P}_n(A_\varepsilon) \geq 1 - \varepsilon$$

and

$$(4.3) \quad \lim_{\delta \rightarrow 0} \sup_{u \in A_\varepsilon} m(u, \delta) = 0.$$

Now we recall the Aldous condition **[A]**, which is connected with condition **[T]**. This condition allows to investigate the modulus of continuity for the sequence of stochastic processes by means of stopped processes.

**Definition 4.4.** A sequence  $(X_n)_{n \in \mathbb{N}}$  satisfies condition **[A]** iff  $\forall \varepsilon > 0, \forall \eta > 0, \exists \delta > 0$  such that for every sequence  $(\tau_n)_{n \in \mathbb{N}}$  of  $\mathbb{F}$ -stopping times with  $\tau_n \leq T$  one has

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq \theta \leq \delta} \mathbb{P} \{\varrho(X_n(\tau_n + \theta), X_n(\tau_n)) \geq \eta\} \leq \varepsilon.$$

**Lemma 4.5.** Conditions **[A]** and **[T]** are equivalent.

**Corollary 4.6** (Tightness criterion). Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of continuous  $\mathbb{F}$ -adapted  $\mathbb{H}$ -valued processes such that

(a) there exists a constant  $C_1 > 0$  such that

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \sup_{s \in [0, T]} \|X_n(s)\|_V^2 \right] \leq C_1,$$

(b) there exists a constant  $C_2 > 0$  such that

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_0^T |X_n(s)|_{D(A)}^2 ds \right] \leq C_2,$$

(c)  $(X_n)_{n \in \mathbb{N}}$  satisfies the Aldous condition [A] in  $H$ .

Let  $\tilde{\mathbb{P}}_n$  be the law of  $X_n$  on  $\mathcal{Z}_T$ . Then for every  $\varepsilon > 0$  there exists a compact subset  $K_\varepsilon$  of  $\mathcal{Z}_T$  such that

$$\sup_{n \in \mathbb{N}} \tilde{\mathbb{P}}_n(K_\varepsilon) \geq 1 - \varepsilon.$$

#### 4.2. The Skorokhod Theorem

We will use the following Jakubowski's version of the Skorokhod Theorem in the form given by Brzeźniak and Ondreját [6], see also [9].

**Theorem 4.7.** *Let  $\mathcal{X}$  be a topological space such that there exists a sequence  $\{f_m\}_{m \in \mathbb{N}}$  of continuous functions  $f_m : \mathcal{X} \rightarrow \mathbb{R}$  that separates points of  $\mathcal{X}$ . Let us denote by  $\mathcal{S}$  the  $\sigma$ -algebra generated by the maps  $\{f_m\}$ . Then*

(a) *every compact subset of  $\mathcal{X}$  is metrizable,*

(b) *if  $(\mu_m)_{m \in \mathbb{N}}$  is a tight sequence of probability measures on  $(\mathcal{X}, \mathcal{S})$ , then there exists a subsequence  $(m_k)_{k \in \mathbb{N}}$ , a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with  $\mathcal{X}$ -valued Borel measurable variables  $\xi_k, \xi$  such that  $\mu_{m_k}$  is the law of  $\xi_k$  and  $\xi_k$  converges to  $\xi$  almost surely on  $\Omega$ . Moreover, the law of  $\xi$  is a Radon measure.*

Using Theorem 4.7, we obtain the following corollary which we will apply to construct a martingale solution to the constrained Navier-Stokes equations.

**Corollary 4.8.** *Let  $(\eta_n)_{n \in \mathbb{N}}$  be a sequence of  $\mathcal{Z}_T$ -valued random variables such that their laws  $\mathcal{L}(\eta_n)$  on  $(\mathcal{Z}_T, \mathcal{T})$  form a tight sequence of probability measures. Then there exists a subsequence  $(n_k)$ , a probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$  and  $\mathcal{Z}_T$ -valued random variables  $\tilde{\eta}, \tilde{\eta}_k, k \in \mathbb{N}$  such that the variables  $\eta_k$  and  $\tilde{\eta}_k$  have the same laws on  $\mathcal{Z}_T$  and  $\tilde{\eta}_k$  converges to  $\tilde{\eta}$  almost surely on  $\tilde{\Omega}$ .*

The proofs of the Lemmas and Theorems stated above can be found in [3, 4] and references therein.

## 5. Existence of solutions

We develop the tools to prove Theorem 3.4 and later prove the theorem in this section.

### 5.1. Faedo-Galerkin approximation and a priori estimates

As mentioned in the introduction, the proof of the existence of a martingale solution is based on the Faedo-Galerkin approximation. In this subsection we first talk about the basic ingredients required for the approximation and then obtain the a priori estimates.

Let  $\{e_i\}_{i=1}^\infty$  be the orthonormal basis in  $H$  composed of eigenvectors of  $A$ . Let  $H_n := \text{span}\{e_1, \dots, e_n\}$  be the subspace with the norm inherited from  $H$ , then  $P_n : H \rightarrow H_n$  given by

$$(5.1) \quad P_n u := \sum_{i=1}^n \langle u, e_i \rangle_H e_i, \quad u \in H.$$

is the orthogonal projection onto  $H_n$ .

Let us consider the classical Faedo-Galerkin approximation of (3.1) in the space  $H_n$ :

$$(5.2) \quad \begin{cases} du_n(t) = - [P_n A u_n(t) + P_n B(u_n(t)) + |\nabla u_n(t)|_{L^2}^2 u_n(t)] dt \\ \quad + \sum_{j=1}^m P_n C_j u_n(t) \circ dW_j(t), & t \in [0, T], \\ u_n(0) = \frac{P_n u_0}{|P_n u_0|}. \end{cases}$$

Using the idea from [7] and the Banach Fixed Point Theorem we can show that the SDE (5.2) has a local maximal solution up to some stopping time  $\tau \leq T$ . In the following lemma we show that this local solution stays on the manifold  $\mathcal{M}$  if we start from the manifold, i.e. if the initial data  $u_n(0) \in \mathcal{M}$  then  $u_n(t) \in \mathcal{M}$  for every  $t \in [0, \tau)$ .

**Lemma 5.1.** *Let  $u_0 \in V \cap \mathcal{M}$  then the solution of (5.2) stays on the manifold  $\mathcal{M}$ , i.e. for all  $t \in [0, \tau)$ ,  $u_n(t) \in \mathcal{M}$ .*

*Proof.* Let  $u_n$  be the solution of (5.2). Then applying Itô formula to the function  $|x|_H^2$  and the process  $u_n$  along (5.2), (2.4) and assumption (A.1), we get

$$\begin{aligned}
\frac{1}{2}d|u_n(t)|_{\mathbf{H}}^2 &= \langle u_n(t), -P_n A u_n(t) - P_n B(u_n(t)) + |\nabla u_n(t)|_{L^2}^2 u_n(t) \rangle_{\mathbf{H}} dt \\
&\quad + \frac{1}{2} \sum_{j=1}^m \langle u_n(t), (P_n C_j)^2 u_n(t) \rangle_{\mathbf{H}} dt + \frac{1}{2} \sum_{j=1}^m \langle P_n C_j u_n(t), P_n C_j u_n(t) \rangle_{\mathbf{H}} dt \\
&\quad + \sum_{j=1}^m \langle u_n(t), P_n C_j u_n(t) dW_j(t) \rangle_{\mathbf{H}} \\
&= -\|u_n(t)\|_{\mathbf{V}}^2 dt + |\nabla u_n(t)|_{L^2}^2 |u_n(t)|_{\mathbf{H}}^2 dt + \frac{1}{2} \sum_{j=1}^m \langle C_j^* u_n(t), C_j u_n(t) \rangle_{\mathbf{H}} dt \\
&\quad + \frac{1}{2} \sum_{j=1}^m |C_j u_n(t)|_{\mathbf{H}}^2 dt \\
&= \|u_n(t)\|_{\mathbf{V}}^2 [|u_n(t)|_{\mathbf{H}}^2 - 1] dt + \frac{1}{2} \sum_{j=1}^m [|C_j u_n(t)|_{\mathbf{H}}^2 - |C_j u_n(t)|_{\mathbf{H}}^2] dt
\end{aligned}$$

thus we get,

$$d [|u_n(t)|_{\mathbf{H}}^2 - 1] = 2\|u_n(t)\|_{\mathbf{V}}^2 [|u_n(t)|_{\mathbf{H}}^2 - 1] dt.$$

Integrating on both sides from 0 to  $t$ , we obtain

$$|u_n(t)|^2 - 1 = [|u_n(0)|_{\mathbf{H}}^2 - 1] \exp \left[ 2 \int_0^t \|u_n(s)\|_{\mathbf{V}}^2 ds \right].$$

Now since  $|u_n(0)|_{\mathbf{H}} = 1$  and  $\int_0^t \|u_n(s)\|_{\mathbf{V}}^2 ds < \infty$ , we get  $|u_n(t)|_{\mathbf{H}} = 1$  for all  $t \in [0, \tau)$ , i.e  $u_n(t) \in \mathcal{M}$  for every  $t \in [0, \tau)$ .  $\square$

Since on the finite dimensional space  $\mathbf{H}_n$  the  $\mathbf{H}$  and  $\mathbf{V}$  norm are equivalent, we can infer from the previous lemma that the  $\mathbf{V}$ -norm of the solution stays bounded. Hence using this non-explosion result as in the case of deterministic setting [1] we can prove the following lemma:

**Lemma 5.2.** *For each  $n \in \mathbb{N}$ , there exists a global solution of (5.2). Moreover for every  $T > 0$ ,  $u_n \in \mathcal{C}([0, T]; \mathbf{H}_n)$ ,  $\mathbb{P}$ -a.s. and for any  $q \in [2, \infty)$*

$$\mathbb{E} \left[ \int_0^T |u_n(s)|_{\mathbf{H}}^q ds \right] < \infty.$$

We will require the following lemma to obtain a priori bounds.

**Lemma 5.3.** *Let  $d = 2$  and  $c : \mathcal{O} \rightarrow \mathbb{R}^2$  is a bounded, divergence free vector field such that  $Ac$  is also bounded. Put*

$$Cu = \Pi((c(x) \cdot \nabla)u).$$

*Then*

$$(5.3) \quad |\langle ACu - CAu, Cu \rangle_H| \leq |Ac|_{L^\infty(\mathcal{O})} |c|_{L^\infty(\mathcal{O})} |\nabla u|_{L^2}^2, \quad u \in H^{3,2}(\mathcal{O}).$$

*Proof.* Let  $c(x) = (c^1, c^2)$  then  $Cu = \Pi(c^1 D_1 + c^2 D_2)u$ . We start by considering  $ACu - CAu$ ,

$$\begin{aligned} ACu - CAu &= -\Pi [\Delta (c^1 D_1 + c^2 D_2) u - (c^1 D_1 + c^2 D_2) \Delta u] \\ &= -\Pi [\Delta(c^1) D_1 + c^1 \Delta D_1 + \Delta(c^2) D_2 + c^2 \Delta D_2 - c^1 \Delta D_1 - c^2 \Delta D_2] u \\ &= -\Pi (\Delta [c^1 D_1 + c^2 D_2] u) = A(c(x) \cdot \nabla)u = A(Cu) \end{aligned}$$

Thus using the Hölder inequality and the Young inequality we get,

$$\begin{aligned} |\langle ACu - CAu, Cu \rangle_H| &= |\langle A(Cu), Cu \rangle_H| \leq |(Ac) \cdot \nabla u|_H |Cu|_H \\ &\leq |Ac|_{L^\infty} |\nabla u|_{L^2} |c|_{L^\infty} |\nabla u|_{L^2} \leq |Ac|_{L^\infty} |c|_{L^\infty} |\nabla u|_{L^2}^2, \end{aligned}$$

hence the proof of (5.3) is complete.  $\square$

**Lemma 5.4.** *Let  $T > 0$  and  $u_n$  be the solution of (5.2). Then under the assumptions (A.1) – (A.2), for all  $\rho > 0$  and  $p \in [1, 1 + \frac{1}{K_c})$ , where  $K_c = \sum_{j=1}^m |c_j|_{L^\infty}^2$ , there exists positive constants  $C_1(p, \rho)$ ,  $C_2(p, \rho)$  and  $C_3(\rho)$  such that*

$$(5.4) \quad \sup_{n \geq 1} \mathbb{E} \left( \sup_{r \in [0, T]} \|u_n(r)\|_V^{2p} \right) \leq C_1(p, \rho),$$

*and*

$$(5.5) \quad \sup_{n \geq 1} \mathbb{E} \int_0^T \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \leq C_2(p, \rho),$$

*provided  $\|u_0\|_V \leq \rho$ . Moreover*

$$(5.6) \quad \sup_{n \geq 1} \mathbb{E} \int_0^T |u_n(s)|_{D(A)}^2 ds \leq C_3(\rho).$$

*Proof.* Let  $u_n(t)$  be the solution of (5.2) then applying the Ito formula to  $\phi(x) = \|x\|_V^2$  and the process  $u_n(t)$ , we get

$$\begin{aligned} d\|u_n(t)\|_V^2 &= 2\langle Au_n(t), -P_nAu_n(t) - P_nB(u_n(t), u_n(t)) + |\nabla u_n(t)|_{L^2}^2 u_n(t) \rangle_H dt \\ &\quad + 2 \times \frac{1}{2} \sum_{j=1}^m \langle Au_n(t), (P_n C_j)^2 u_n(t) \rangle_H dt + 2 \times \frac{1}{2} \sum_{j=1}^m \langle AP_n C_j u_n(t), P_n C_j u_n(t) \rangle_H dt \\ &\quad + 2 \sum_{j=1}^m \langle Au_n(t), P_n C_j u_n(t) dW_j(t) \rangle_H. \end{aligned}$$

Now since  $\langle |\nabla u_n(t)|_{L^2}^2 u_n(t), Au_n(t) - |\nabla u_n(t)|_{L^2}^2 u_n(t) \rangle = 0$ , using (2.4) and assumption (A.1), we have

$$\begin{aligned} d\|u_n(t)\|_V^2 &= -2\langle Au_n(t) - |\nabla u_n(t)|_{L^2}^2 u_n(t), Au_n(t) - |\nabla u_n(t)|_{L^2}^2 u_n(t) \rangle_H dt \\ &\quad + 2\langle |\nabla u_n(t)|_{L^2}^2 u_n(t), Au_n(t) - |\nabla u_n(t)|_{L^2}^2 u_n(t) \rangle_H dt \\ &\quad - 2\langle Au_n(t), B(u_n(t), u_n(t)) \rangle_H dt + \sum_{j=1}^m \langle Au_n(t), C_j^2 u_n(t) \rangle_H dt \\ &\quad + \sum_{j=1}^m \langle AC_j u_n(t), C_j u_n(t) \rangle_H dt + 2 \sum_{j=1}^m \langle Au_n(t), C_j u_n(t) dW_j(t) \rangle_H \\ &= -2\langle Au_n(t) - |\nabla u_n(t)|_{L^2}^2 u_n(t), Au_n(t) - |\nabla u_n(t)|_{L^2}^2 u_n(t) \rangle_H dt + \sum_{j=1}^m \langle AC_j u_n(t) - C_j Au_n(t), C_j u_n(t) \rangle_H dt \\ &\quad + 2 \sum_{j=1}^m \langle Au_n(t), C_j u_n(t) dW_j(t) \rangle_H. \end{aligned}$$

Now integrating on both sides and using Assumption (A.1) and Lemma 5.3, we get

$$\begin{aligned} (5.7) \quad &\|u_n(t)\|_V^2 + 2 \int_0^t \|Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)\|_H^2 ds \\ &= \|u_n(0)\|_V^2 + \sum_{j=1}^m \int_0^t \langle AC_j u_n(s) - C_j Au_n(s), C_j u_n(s) \rangle_H ds \\ &\quad + 2 \sum_{j=1}^m \int_0^t \langle Au_n(s), C_j u_n(s) dW_j(s) \rangle_H \end{aligned}$$

$$(5.8) \quad \leq \|u_n(0)\|_V^2 + K \int_0^t \|\nabla u_n(s)\|_{L^2}^2 ds + 2 \sum_{j=1}^m \int_0^t \langle Au_n(s), C_j u_n(s) dW_j(s) \rangle_H,$$



where  $K = \sum_{j=1}^m |Ac_j|_{L^\infty} |c_j|_{L^\infty}$ .

By Lemma 5.2, we infer that the process

$$\mu_n(t) = \sum_{j=1}^m \int_0^t \langle Au_n(s), C_j u_n(s) dW_j(s) \rangle_H, \quad t \in [0, T]$$

is a martingale and that  $\mathbb{E}[\mu_n(t)] = 0$ . Thus

$$\begin{aligned} \mathbb{E}\|u_n(t)\|_V^2 + 2\mathbb{E} \int_0^t |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\ (5.9) \quad \leq \mathbb{E}\|u_n(0)\|_V^2 + K \int_0^t \mathbb{E}\|u_n(s)\|_V^2 ds. \end{aligned}$$

Applying the Grönwall Lemma, we have

$$\mathbb{E}\|u_n(t)\|_V^2 \leq \mathbb{E}\|u_n(0)\|_V^2 e^{Kt}.$$

Thus

$$(5.10) \quad \sup_{n \geq 1} \sup_{t \in [0, T]} \mathbb{E}\|u_n(t)\|_V^2 \leq \mathbb{E}\|u(0)\|_V^2 e^{KT}.$$

Note that using (5.10) in (5.9), we also have the following estimate

$$(5.11) \quad \sup_{n \geq 1} \mathbb{E} \int_0^T |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \leq \mathbb{E}\|u(0)\|_V^2 e^{KT}.$$

Let  $\xi(t) = \|u_n(t)\|_V^{2p}$ ,  $t \in [0, T]$  and  $\phi(x) = x^p$ , for some fixed  $p \in [1, \infty)$ . Using the Itô formula and (5.8), we obtain

$$\begin{aligned} \|u_n(t)\|_V^{2p} &= \|u_n(0)\|_V^{2p} - 2p \int_0^t \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\ &\quad + p \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-1)} \langle AC_j u_n(s) - C_j Au_n(s), C_j u_n(s) \rangle_H ds \\ &\quad + 2p(p-1) \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-2)} \langle Au_n(s), C_j u_n(s) \rangle_H^2 ds \\ (5.12) \quad &\quad + 2p \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-1)} \langle Au_n(s), C_j u_n(s) dW_j(s) \rangle_H \end{aligned}$$

From Assumption (A.1),  $\langle Cu_n(s), u_n(s) \rangle = 0$  and using Lemma 5.3 with the same  $K$  as before we get

$$\begin{aligned}
& \|u_n(t)\|_V^{2p} + 2p \int_0^t \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\
& \leq \|u_n(0)\|_V^{2p} + pK \int_0^t \|u_n(s)\|_V^{2(p-1)} |\nabla u_n(s)|_{L^2}^2 ds \\
& \quad + 2p \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-1)} \langle Au_n(s), C_j u_n(s) dW_j(s) \rangle_H \\
& \quad + 2p(p-1) \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-2)} \langle Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s), C_j u_n(s) \rangle_H^2 ds.
\end{aligned}$$

Now since  $\|u\|_V := |\nabla u|_{L^2}$ , we have

$$\begin{aligned}
& \|u_n(t)\|_V^{2p} + 2p \int_0^t \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\
& \leq \|u_n(0)\|_V^{2p} + pK \int_0^t \|u_n(s)\|_V^{2p} ds \\
& \quad + 2p(p-1) \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-2)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 |C_j u_n(s)|_H^2 ds \\
& \quad + 2p \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-1)} \langle Au_n(s), C_j u_n(s) dW_j(s) \rangle_H.
\end{aligned}$$

On rearranging we get

$$\begin{aligned}
& \|u_n(t)\|_V^{2p} + 2p \int_0^t \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\
& \leq \|u_n(0)\|_V^{2p} + pK \int_0^t \|u_n(s)\|_V^{2p} ds \\
& \quad + 2p(p-1)K_c \int_0^t \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\
& \quad + 2p \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-1)} \langle Au_n(s), C_j u_n(s) dW_j(s) \rangle_H,
\end{aligned}$$

where  $K_c$  is the positive constant defined in the statement of Lemma. For  $p \in [1, 1 + \frac{1}{K_c})$ ,  $K_p = 2p[1 - K_c(p-1)] > 0$ , thus

$$\begin{aligned}
& \|u_n(t)\|_V^{2p} + K_p \int_0^t \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\
(5.13) \quad & \leq \|u_n(0)\|_V^{2p} + pK \int_0^t \|u_n(s)\|_V^{2p} ds \\
& \quad + 2p \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-1)} \langle Au_n(s), C_j u_n(s) dW_j(s) \rangle_H.
\end{aligned}$$

Using Lemma 5.2 we infer that the process

$$\eta_n(t) = \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-1)} \langle Au_n(s), C_j u_n(s) dW_j(s) \rangle_H, \quad t \in [0, T],$$

is a martingale and  $\mathbb{E}[\eta_n(t)] = 0$ . Thus

$$\begin{aligned}
& \mathbb{E}\|u_n(t)\|_V^{2p} + K_p \mathbb{E} \int_0^t \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\
(5.14) \quad & \leq \mathbb{E}\|u_n(0)\|_V^{2p} + pK \mathbb{E} \int_0^t \|u_n(s)\|_V^{2p} ds.
\end{aligned}$$

In particular

$$\mathbb{E}\|u_n(t)\|_V^{2p} \leq \mathbb{E}\|u_n(0)\|_V^{2p} + pK \mathbb{E} \int_0^t \|u_n(s)\|_V^{2p} ds,$$

thus by the Grönwall Lemma

$$(5.15) \quad \mathbb{E}\|u_n(t)\|_V^{2p} \leq \mathbb{E}\|u_n(0)\|_V^{2p} e^{Kpt}.$$

Hence

$$(5.16) \quad \sup_{n \geq 1} \sup_{t \in [0, T]} \mathbb{E}\|u_n(t)\|_V^{2p} \leq \mathbb{E}\|u_0\|_V^{2p} e^{KpT}$$

Note that using (5.16) in (5.14), we also have the following estimate,

$$(5.17) \quad \sup_{n \geq 1} \mathbb{E} \int_0^T \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \leq \mathbb{E}\|u_0\|_V^{2p} e^{KpT}.$$

In order to prove (5.4) we start from (5.12),

$$\begin{aligned}
\|u_n(t)\|_V^{2p} &= \|u_n(0)\|_V^{2p} - 2p \int_0^t \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\
&\quad + p \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-1)} \langle AC_j u_n(s) - C_j Au_n(s), C_j u_n(s) \rangle_H ds \\
&\quad + 2p(p-1) \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-2)} \langle Au_n(s), C_j u_n(s) \rangle_H^2 ds \\
&\quad + 2p \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-1)} \langle Au_n(s), C_j u_n(s) dW_j(s) \rangle_H.
\end{aligned}$$

From assumption (A.1),  $\langle C_j u_n(s), u_n(s) \rangle_H = 0$  for every  $j$  hence,

$$\begin{aligned}
\|u_n(t)\|_V^{2p} &+ 2p \int_0^t \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds = \|u_n(0)\|_V^{2p} \\
&\quad + p \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-1)} \langle AC_j u_n(s) - C_j Au_n(s), C_j u_n(s) \rangle_H ds \\
&\quad + 2p(p-1) \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-2)} \langle Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s), C_j u_n(s) \rangle_H^2 ds \\
&\quad + 2p \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-1)} \langle Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s), C_j u_n(s) dW_j(s) \rangle_H.
\end{aligned}$$

Using Lemma 5.3, we obtain

$$\begin{aligned}
\mathbb{E} \sup_{r \in [0, t]} &\left[ \|u_n(r)\|_V^{2p} + 2p \int_0^r \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \right] \\
&\leq \mathbb{E} \|u_n(0)\|_V^{2p} + pK \mathbb{E} \sup_{r \in [0, t]} \left[ \int_0^r \|u_n(s)\|_V^{2(p-1)} |\nabla u_n(s)|_{L^2}^2 ds \right] \\
&\quad + 2p(p-1) \mathbb{E} \sup_{r \in [0, t]} \left[ \sum_{j=1}^m \int_0^r \|u_n(s)\|_V^{2(p-2)} \langle Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s), C_j u_n(s) \rangle_H^2 ds \right] \\
&\quad + 2p \mathbb{E} \sup_{r \in [0, t]} \left[ \sum_{j=1}^m \int_0^r \|u_n(s)\|_V^{2(p-1)} \langle Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s), C_j u_n(s) dW_j(s) \rangle_H \right].
\end{aligned}$$

Using the Hölder inequality, we have

$$\begin{aligned}
& \mathbb{E} \sup_{r \in [0, t]} \|u_n(r)\|_V^{2p} + 2p \mathbb{E} \sup_{r \in [0, t]} \int_0^r \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\
& \leq \mathbb{E} \|u_n(0)\|_V^{2p} + pK \mathbb{E} \sup_{r \in [0, t]} \left[ \int_0^r \|u_n(s)\|_V^{2(p-1)} \|u_n(s)\|_V^2 ds \right] \\
& \quad + 2p(p-1)K_c \mathbb{E} \sup_{r \in [0, t]} \left[ \int_0^r \|u_n(s)\|_V^{2(p-2)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 |\nabla u_n(s)|_{L^2}^2 ds \right] \\
& \quad + 2p \mathbb{E} \sup_{r \in [0, t]} \left[ \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-1)} \langle Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s), C_j u_n(s) dW_j(s) \rangle_H \right].
\end{aligned}$$

On rearranging we get

$$\begin{aligned}
& \mathbb{E} \sup_{r \in [0, t]} \|u_n(r)\|_V^{2p} + 2p \mathbb{E} \sup_{r \in [0, t]} \int_0^r \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\
& \leq \mathbb{E} \|u_n(0)\|_V^{2p} + pK \mathbb{E} \sup_{r \in [0, t]} \int_0^r \|u_n(s)\|_V^{2p} ds \\
& \quad + 2p(p-1)K_c \mathbb{E} \sup_{r \in [0, t]} \int_0^r \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\
(5.18) \quad & + 2p \mathbb{E} \sup_{r \in [0, t]} \left| \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{2(p-1)} \langle Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s), C_j u_n(s) dW_j(s) \rangle_H \right|.
\end{aligned}$$

Using the Burkholder- Davis- Gundy inequality, we get

$$\begin{aligned}
& \mathbb{E} \sup_{r \in [0, t]} \left| \sum_{j=1}^m \int_0^r \|u_n(s)\|_V^{2(p-1)} \langle Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s), C_j u_n(s) dW_j(s) \rangle_H \right| \\
& \leq 3 \mathbb{E} \left| \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{4(p-1)} \langle Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s), C_j u_n(s) \rangle_H^2 ds \right|^{1/2} \\
& \leq 3 \mathbb{E} \left| \sum_{j=1}^m \int_0^t \|u_n(s)\|_V^{4(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 |C_j u_n(s)|_H^2 ds \right|^{1/2} \\
& \leq 3 \mathbb{E} \sqrt{K_c} \left[ \int_0^t \|u_n(s)\|_V^{2p} \|u_n(s)\|_V^{2(p-1)} |Au_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \right]^{1/2}.
\end{aligned}$$

Using the Hölder inequality and the Young inequality, we obtain

$$\begin{aligned}
& \mathbb{E} \sup_{r \in [0, t]} \left| \sum_{j=1}^m \int_0^r \|u_n(s)\|_V^{2(p-1)} \langle \mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s), C_j u_n(s) dW_j(s) \rangle_H \right| \\
& \leq 3\mathbb{E} \left[ \sqrt{K_c} \left( \sup_{r \in [0, t]} \|u_n(r)\|_V^{2p} \right)^{1/2} \left( \int_0^t \|u_n(s)\|_V^{2(p-1)} |\mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \right)^{1/2} \right] \\
& \leq 3\mathbb{E} \left[ \varepsilon \sup_{r \in [0, t]} \|u_n(r)\|_V^{2p} + \frac{K_c}{4\varepsilon} \int_0^t \|u_n(s)\|_V^{2(p-1)} |\mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \right].
\end{aligned}$$

Thus using this in (5.18), we get

$$\begin{aligned}
& \mathbb{E} \sup_{r \in [0, t]} \|u_n(r)\|_V^{2p} + 2\mathbb{E} \sup_{r \in [0, t]} \int_0^r \|u_n(s)\|_V^{2(p-1)} |\mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\
& \leq \mathbb{E} \|u_n(0)\|_V^{2p} + pK\mathbb{E} \sup_{r \in [0, t]} \int_0^r \|u_n(s)\|_V^{2p} ds + 6p\varepsilon \mathbb{E} \sup_{r \in [0, t]} \|u_n(r)\|_V^{2p} \\
& \quad + 2p(p-1)K_c\mathbb{E} \sup_{r \in [0, t]} \int_0^r \|u_n(s)\|_V^{2(p-1)} |\mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\
(5.19) \quad & \quad + \frac{3pK_c}{2\varepsilon} \mathbb{E} \int_0^t \|u_n(s)\|_V^{2(p-1)} |\mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds
\end{aligned}$$

Hence for  $\varepsilon = \frac{1}{12p}$ , Eq. (5.19) reduces to

$$\begin{aligned}
& \mathbb{E} \sup_{r \in [0, t]} \|u_n(r)\|_V^{2p} + 4\mathbb{E} \sup_{r \in [0, t]} \int_0^r \|u_n(s)\|_V^{2(p-1)} |\mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\
& \leq 2\mathbb{E} \|u_n(0)\|_V^{2p} + 2pK\mathbb{E} \sup_{r \in [0, t]} \int_0^r \|u_n(s)\|_V^{2p} ds \\
& \quad + 4p(p-1)K_c\mathbb{E} \sup_{r \in [0, t]} \int_0^r \|u_n(s)\|_V^{2(p-1)} |\mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\
& \quad + 36p^2K_c\mathbb{E} \int_0^t \|u_n(s)\|_V^{2(p-1)} |\mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds.
\end{aligned}$$

Since  $\int_0^r |\mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds$  is an increasing function, we have

$$\begin{aligned} \mathbb{E} \sup_{r \in [0, t]} \|u_n(r)\|_V^{2p} + 4\mathbb{E} \int_0^t \|u_n(s)\|_V^{2(p-1)} |\mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds \\ \leq 2\mathbb{E}\|u_n(0)\|_V^{2p} + 2pK\mathbb{E} \sup_{r \in [0, t]} \int_0^r \|u_n(s)\|_V^{2p} ds \\ + 4pK_c [10p - 1] \mathbb{E} \int_0^t \|u_n(s)\|_V^{2(p-1)} |\mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds. \end{aligned}$$

In particular

$$\begin{aligned} \mathbb{E} \sup_{r \in [0, t]} \|u_n(r)\|_V^{2p} \leq 2\mathbb{E}\|u_n(0)\|_V^{2p} + 2pK\mathbb{E} \int_0^t \sup_{r \in [0, s]} \|u_n(r)\|_V^{2p} ds \\ + 4pK_c [10p - 1] \mathbb{E} \int_0^t \|u_n(s)\|_V^{2(p-1)} |\mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds. \end{aligned}$$

Thus by the Grönwall Lemma, we get

$$\mathbb{E} \sup_{r \in [0, t]} \|u_n(r)\|_V^{2p} \leq K_3 \exp(K_4 t),$$

where

$$\begin{aligned} K_3 &= 2\mathbb{E}\|u_n(0)\|_V^{2p} + 4p[10p - 1]K_c\mathbb{E} \int_0^t \|u_n(s)\|_V^{2(p-1)} |\mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds, \\ K_4 &= 2pK. \end{aligned}$$

Since  $\mathbb{E}\|u_n(0)\|_V^{2p} \leq \mathbb{E}\|u_0\|_V^{2p}$  and using (5.17), for  $p \in [1, 1 + \frac{1}{K_c})$

$$\mathbb{E} \int_0^T \|u_n(s)\|_V^{2(p-1)} |\mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds$$

is uniformly bounded in  $n$ , thus

$$\sup_{n \geq 1} \mathbb{E} \sup_{r \in [0, T]} \|u_n(r)\|_V^{2p} \leq C_1(p, \rho).$$

Now we will establish (5.6). Note that

$$\mathbb{E} \int_0^T |u_n(s)|_{D(A)}^2 ds = \mathbb{E} \int_0^T |\mathbf{A}u_n(s) - |\nabla u_n(s)|_{L^2}^2 u_n(s)|_H^2 ds + \mathbb{E} \int_0^T \|u_n(s)\|^4 ds.$$

Using (5.4) for  $p = 2$  and (5.5) for  $p = 1$ , we get

$$\sup_{n \geq 1} \mathbb{E} \int_0^T |u_n(s)|_{D(A)}^2 ds \leq C_2(1, \rho) + C_1(2, \rho)T =: C_3(\rho).$$

□

## 5.2. Tightness

In this subsection using the a priori estimates from the Lemma 5.4 and the Corollary 4.6 we will prove that for every  $n \in \mathbb{N}$  the measures  $\mathcal{L}(u_n)$  on  $(\mathcal{Z}_T, \mathcal{T})$  defined by the solution of the stochastic ODE (5.2) are tight. The following is the main result of this subsection.

**Lemma 5.5.** *The set of measures  $\{\mathcal{L}(u_n), n \in \mathbb{N}\}$  is tight on  $(\mathcal{Z}_T, \mathcal{T})$ .*

*Proof.* We apply Corollary 4.6. According to the a priori estimates (5.4) (for  $p = 1$ ) and (5.6), conditions (a) and (b) of Corollary 4.6 are satisfied. Thus it is sufficient to prove that the sequence  $(u_n)_{n \in \mathbb{N}}$  satisfies the Aldous condition [A] in  $H$ . Let  $(\tau_n)_{n \in \mathbb{N}}$  be a sequence of stopping times such that  $0 \leq \tau_n \leq T$ . By (5.2), for  $t \in [0, T]$  we have

$$\begin{aligned} u_n(t) &= u_n(0) - \int_0^t P_n A u_n(s) ds - \int_0^t P_n B(u_n(s)) ds + \int_0^t |\nabla u_n(s)|_{L^2}^2 u_n(s) ds \\ &\quad + \frac{1}{2} \int_0^t (P_n C)^2 u_n(s) ds + \int_0^t P_n C u_n(s) dW(s) \\ &:= J_1^n + J_2^n(t) + J_3^n(t) + J_4^n(t) + J_5^n(t) + J_6^n(t), \quad t \in [0, T]. \end{aligned}$$

Let  $\theta > 0$ . First we make some estimates for each term of the above equality.

**Ad.  $J_2^n$ .** Since  $A : D(A) \rightarrow H$ , then by the Hölder inequality and (5.6), we have the following estimates

$$\begin{aligned} \mathbb{E} [|J_2^n(\tau_n + \theta) - J_2^n(\tau_n)|_H] &= \mathbb{E} \left| \int_{\tau_n}^{\tau_n + \theta} P_n A u_n(s) ds \right|_H \leq c \mathbb{E} \int_{\tau_n}^{\tau_n + \theta} |A u_n(s)|_H ds \\ (5.20) \quad &\leq c \mathbb{E} \int_{\tau_n}^{\tau_n + \theta} |u_n(s)|_{D(A)} ds \leq c \theta^{\frac{1}{2}} \left( \mathbb{E} \left[ \int_0^T |u_n(s)|_{D(A)}^2 ds \right] \right)^{\frac{1}{2}} \leq c C_3^{\frac{1}{2}} \cdot \theta^{\frac{1}{2}} =: c_2 \cdot \theta^{\frac{1}{2}}. \end{aligned}$$

**Ad.  $J_3^n$ .** Since  $B : V \times V \rightarrow H$  is bilinear and continuous, then using (2.3), the Cauchy Schwartz inequality, (5.4) and (5.6), we have the following estimates



$$\begin{aligned}
\mathbb{E}[|J_3^n(\tau_n + \theta) - J_3^n(\tau_n)|_{\mathbb{H}}] &= \mathbb{E} \left| \int_{\tau_n}^{\tau_n + \theta} P_n B(u_n(s)) ds \right|_{\mathbb{H}} \leq c \mathbb{E} \int_{\tau_n}^{\tau_n + \theta} |B(u_n(s), u_n(s))|_{\mathbb{H}} ds \\
&\leq c \mathbb{E} \int_{\tau_n}^{\tau_n + \theta} |u_n(s)|_{\mathbb{H}}^{\frac{1}{2}} \|u_n(s)\|_{\mathbb{V}} |u_n(s)|_{\mathbb{D}(\mathbb{A})}^{\frac{1}{2}} ds \leq c \mathbb{E} \left[ \int_{\tau_n}^{\tau_n + \theta} \|u_n(s)\|_{\mathbb{V}}^{\frac{3}{2}} |u_n(s)|_{\mathbb{D}(\mathbb{A})}^{1/2} ds \right] \\
&\leq c \mathbb{E} \left( \left[ \int_{\tau_n}^{\tau_n + \theta} \|u_n(s)\|_{\mathbb{V}}^2 ds \right]^{\frac{3}{4}} \left[ \int_{\tau_n}^{\tau_n + \theta} |u_n(s)|_{\mathbb{D}(\mathbb{A})}^2 ds \right]^{\frac{1}{4}} \right) \\
(5.21) \quad &\leq c \theta^{\frac{3}{4}} \left[ \mathbb{E} \sup_{s \in [0, T]} \|u_n(s)\|_{\mathbb{V}}^2 \right]^{\frac{3}{4}} \left[ \mathbb{E} \int_0^T |u_n(s)|_{\mathbb{D}(\mathbb{A})}^2 ds \right]^{\frac{1}{4}} \leq c C_1(1)^{\frac{3}{4}} C_3^{\frac{1}{4}} \cdot \theta^{\frac{3}{4}} =: c_3 \cdot \theta^{\frac{3}{4}}.
\end{aligned}$$

**Ad.  $J_4^n$ .** Using Lemma 5.1 and estimate (5.4), we have

$$\begin{aligned}
\mathbb{E}[|J_4^n(\tau_n + \theta) - J_4^n(\tau_n)|_{\mathbb{H}}] &= \mathbb{E} \left| \int_{\tau_n}^{\tau_n + \theta} |\nabla u_n(s)|_{L^2}^2 u_n(s) ds \right|_{\mathbb{H}} \\
(5.22) \quad &\leq \mathbb{E} \int_{\tau_n}^{\tau_n + \theta} |\nabla u_n(s)|_{L^2}^2 |u_n(s)|_{\mathbb{H}} ds \leq \mathbb{E} \sup_{s \in [0, T]} \|u_n(s)\|_{\mathbb{V}}^2 \theta \leq C_1(1) \cdot \theta =: c_4 \cdot \theta.
\end{aligned}$$

**Ad.  $J_5^n$ .** Since  $C$  is linear and continuous, then using the Cauchy-Schwartz inequality, Assumption (A.1) and (5.6), we have the following

$$\begin{aligned}
\mathbb{E}[|J_5^n(\tau_n + \theta) - J_5^n(\tau_n)|_{\mathbb{H}}] &= \mathbb{E} \left| \frac{1}{2} \sum_{j=1}^m \int_{\tau_n}^{\tau_n + \theta} (P_n C_j)^2 u_n(s) ds \right|_{\mathbb{H}} \\
&\leq \frac{1}{2} c \mathbb{E} \left( \sum_{j=1}^m \int_{\tau_n}^{\tau_n + \theta} |C_j^2 u_n(s)|_{\mathbb{H}} ds \right) \leq \frac{1}{2} c K_c \mathbb{E} \int_{\tau_n}^{\tau_n + \theta} |u_n(s)|_{\mathbb{D}(\mathbb{A})} ds \\
(5.23) \quad &\leq \frac{1}{2} c K_c \left[ \mathbb{E} \int_0^T |u_n(s)|_{\mathbb{D}(\mathbb{A})}^2 ds \right]^{\frac{1}{2}} \theta^{\frac{1}{2}} \leq \frac{c K_c}{2} C_3^{\frac{1}{2}} \cdot \theta^{\frac{1}{2}} =: c_5 \cdot \theta^{\frac{1}{2}}.
\end{aligned}$$

**Ad.  $J_6^n$ .** Using the Ito symmetry, Assumption (A.1) and estimate (5.4), we obtain the following

$$\begin{aligned}
\mathbb{E}[|J_6^n(\tau_n + \theta) - J_6^n(\tau_n)|_{\mathbb{H}}^2] &= \mathbb{E} \left| \int_{\tau_n}^{\tau_n + \theta} P_n C u_n(s) dW(s) \right|_{\mathbb{H}}^2 \leq c \mathbb{E} \int_{\tau_n}^{\tau_n + \theta} |C u_n(s)|_{\mathbb{H}}^2 ds \\
(5.24) \quad &\leq c K_1 \mathbb{E} \int_{\tau_n}^{\tau_n + \theta} \|u_n(s)\|_{\mathbb{V}}^2 ds \leq c K_1 \mathbb{E} \sup_{s \in [0, T]} \|u_n(s)\|_{\mathbb{V}}^2 \theta \leq c K_1 C_1(1) \cdot \theta =: c_6 \cdot \theta.
\end{aligned}$$

Let us fix  $\kappa > 0$  and  $\varepsilon > 0$ . By the Chebyshev's inequality and estimates (5.20) - (5.23), we obtain

$$\mathbb{P}(\{|J_i^n(\tau_n + \theta) - J_i^n(\tau_n)|_{\mathbb{H}} \geq \kappa\}) \leq \frac{1}{\kappa} \mathbb{E}[|J_i^n(\tau_n + \theta) - J_i^n(\tau_n)|_{\mathbb{H}}] \leq \frac{c_i \theta}{\kappa}; \quad n \in \mathbb{N},$$

where  $i = 1, \dots, 5$ . Let  $\delta_i = \frac{\kappa}{c_i} \varepsilon$ . Then

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq \theta \leq \delta_i} \mathbb{P}(\{|J_i^n(\tau_n + \theta) - J_i^n(\tau_n)|_{\mathbb{H}} \geq \kappa\}) \leq \varepsilon, \quad i = 1 \dots 5.$$

By the Chebyshev inequality and (5.24), we have

$$\begin{aligned} \mathbb{P}(\{|J_6^n(\tau_n + \theta) - J_6^n(\tau_n)|_{\mathbb{H}} \geq \kappa\}) &\leq \frac{1}{\kappa^2} \mathbb{E}[|J_6^n(\tau_n + \theta) - J_6^n(\tau_n)|_{\mathbb{H}}^2] \\ &\leq \frac{c_6 \theta}{\kappa^2}, \quad n \in \mathbb{N}. \end{aligned}$$

Let  $\delta_6 = \frac{\kappa^2}{C_6} \varepsilon$ . Then

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq \theta \leq \delta_6} \mathbb{P}(\{|J_6^n(\tau_n + \theta) - J_6^n(\tau_n)|_{\mathbb{H}} \geq \kappa\}) \leq \varepsilon.$$

Since [A] holds for each term  $J_i^n$ ,  $i = 1, 2, \dots, 6$ ; we infer that it holds also for  $(u_n)$ .  $\square$

### 5.3. Proof of Theorem 3.4

By Lemma 5.5 and the Prohorov Theorem the set of measures  $\{\mathcal{L}(u_n), n \in \mathbb{N}\}$  is tight on the space  $(\mathcal{Z}_T, \mathcal{T})$  defined by (4.1). Hence by Corollary 4.8 there exist a subsequence  $(n_k)_{k \in \mathbb{N}}$ , a probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$  and, on this space,  $\mathcal{Z}_T$ -valued random variables  $\tilde{u}, \tilde{u}_{n_k}, k \geq 1$  such that

$$(5.25) \quad \tilde{u}_{n_k} \text{ has the same law as } u_{n_k} \text{ and } \tilde{u}_{n_k} \rightarrow \tilde{u} \text{ in } \mathcal{Z}_T, \quad \tilde{\mathbb{P}} - \text{a.s.}$$

$\tilde{u}_{n_k} \rightarrow \tilde{u}$  in  $\mathcal{Z}_T, \tilde{\mathbb{P}} - \text{a.s.}$  precisely means that

$$\begin{aligned} \tilde{u}_{n_k} &\rightarrow \tilde{u} \text{ in } \mathcal{C}([0, T]; \mathbb{H}), \\ \tilde{u}_{n_k} &\rightharpoonup \tilde{u} \text{ in } L^2(0, T; \mathbb{D}(\mathbb{A})), \\ \tilde{u}_{n_k} &\rightarrow \tilde{u} \text{ in } L^2(0, T; \mathbb{V}), \\ \tilde{u}_{n_k} &\rightarrow \tilde{u} \text{ in } \mathcal{C}([0, T]; \mathbb{V}_w). \end{aligned}$$

Let us denote the subsequence  $(\tilde{u}_{n_k})$  again by  $(\tilde{u}_n)_{n \in \mathbb{N}}$ .

Since  $u_n \in \mathcal{C}([0, T]; H_n), \mathbb{P}$ -a.s. and  $\mathcal{C}([0, T]; H_n)$  is a Borel subset of  $\mathcal{C}([0, T]; H) \cap L^2(0, T; V)$  and also  $\tilde{u}_n, u_n$  have the same laws on  $\mathcal{Z}_T$  we can make the following inferences

$$\begin{aligned} \mathcal{L}(\tilde{u}_n)(\mathcal{C}([0, T]; H_n) &= 1, \quad n \geq 1, \\ |\tilde{u}_n(t)|_H &= |u_n(t)|_H, \quad a.s. \end{aligned}$$

Also from (5.25)  $\tilde{u}_n \rightarrow \tilde{u}$  in  $\mathcal{C}([0, T]; H)$  and by Lemma 5.1  $u_n(t) \in \mathcal{M}$  for every  $t \in [0, T]$ . Therefore we can conclude that

$$(5.26) \quad \tilde{u}(t) \in \mathcal{M}, \quad t \in [0, T].$$

Moreover by (5.4) and (5.6), for  $p \in [1, 1 + \frac{1}{K_c})$

$$(5.27) \quad \sup_{n \in \mathbb{N}} \tilde{\mathbb{E}} \left( \sup_{0 \leq s \leq T} \|\tilde{u}_n(s)\|_V^{2p} \right) \leq C_1(p),$$

$$(5.28) \quad \sup_{n \in \mathbb{N}} \tilde{\mathbb{E}} \left[ \int_0^T |\tilde{u}_n(s)|_{D(A)}^2 ds \right] \leq C_3.$$

By inequality (5.28) we infer that the sequence  $(\tilde{u}_n)$  contains a subsequence, still denoted by  $(\tilde{u}_n)$  convergent weakly in the space  $L^2([0, T] \times \tilde{\Omega}; D(A))$ . Since by (5.25)  $\tilde{\mathbb{P}}$ -a.s  $\tilde{u}_n \rightarrow \tilde{u}$  in  $\mathcal{Z}_T$ , we conclude that  $\tilde{u} \in L^2([0, T] \times \tilde{\Omega}; D(A))$ , i.e.

$$(5.29) \quad \tilde{\mathbb{E}} \left[ \int_0^T |\tilde{u}(s)|_{D(A)}^2 ds \right] < \infty.$$

Similarly by inequality (5.27) we can choose a subsequence of  $(\tilde{u}_n)$  convergent weak star in the space  $L^2(\tilde{\Omega}; L^\infty(0, T; V))$  and, using (5.25), we infer that

$$(5.30) \quad \tilde{\mathbb{E}} \left( \sup_{0 \leq s \leq T} \|\tilde{u}(s)\|_V^2 \right) < \infty.$$

For each  $n \geq 1$ , let us consider a process  $\tilde{M}_n$  with trajectories in  $\mathcal{C}([0, T]; H_n)$ , in particular in  $\mathcal{C}([0, T]; H)$  defined by

$$\begin{aligned} \tilde{M}_n(t) &= \tilde{u}_n(t) - P_n \tilde{u}(0) + \int_0^t P_n A \tilde{u}_n(s) ds + \int_0^t P_n B(\tilde{u}_n(s)) ds \\ (5.31) \quad &- \int_0^t |\nabla \tilde{u}_n(s)|^2 \tilde{u}_n(s) ds - \frac{1}{2} \sum_{j=1}^m \int_0^t (P_n C_j)^2 \tilde{u}_n(s) ds \quad t \in [0, T]. \end{aligned}$$

**Lemma 5.6.**  $\tilde{M}_n$  is a square integrable martingale with respect to the filtration  $\tilde{\mathbb{F}}_n = (\tilde{\mathcal{F}}_{n,t})$ , where  $\tilde{\mathcal{F}}_{n,t} = \sigma\{\tilde{u}_n(s), s \leq t\}$  with the quadratic variation

$$(5.32) \quad \langle \tilde{M}_n \rangle_t = \int_0^t \sum_{j=1}^m |P_n C_j \tilde{u}_n(s)|_{\mathbb{H}}^2 ds.$$

*Proof.* Indeed since  $\tilde{u}_n$  and  $u_n$  have the same laws, for all  $s, t \in [0, T]$ ,  $s \leq t$ , for all bounded continuous functions  $h$  on  $\mathcal{C}([0, s]; \mathbb{H})$ , and all  $\psi, \zeta \in \mathbb{H}$ , we have

$$(5.33) \quad \tilde{\mathbb{E}} \left[ \langle \tilde{M}_n(t) - \tilde{M}_n(s), \psi \rangle_{\mathbb{H}} h(\tilde{u}_{n|[0,s]}) \right] = 0$$

and

$$(5.34) \quad \begin{aligned} & \tilde{\mathbb{E}} \left[ \left( \langle \tilde{M}_n(t), \psi \rangle_{\mathbb{H}} \langle \tilde{M}_n(t), \zeta \rangle_{\mathbb{H}} - \langle \tilde{M}_n(s), \psi \rangle_{\mathbb{H}} \langle \tilde{M}_n(s), \zeta \rangle_{\mathbb{H}} \right. \right. \\ & \left. \left. - \sum_{j=1}^m \int_s^t \langle (C_j \tilde{u}_n(\sigma))^* P_n \psi, (C_j \tilde{u}_n(\sigma))^* P_n \zeta \rangle_{\mathbb{R}} d\sigma \right) \cdot h(\tilde{u}_{n|[0,s]}) \right] = 0. \end{aligned}$$

□

**Lemma 5.7.** Let us define a process  $\tilde{M}$  for  $t \in [0, T]$  by

$$(5.35) \quad \begin{aligned} \tilde{M}(t) &= \tilde{u}(t) - \tilde{u}(0) + \int_0^t A \tilde{u}(s) ds + \int_0^t B(\tilde{u}(s)) ds \\ &\quad - \int_0^t |\nabla \tilde{u}(s)|_{L^2}^2 \tilde{u}(s) ds - \frac{1}{2} \sum_{j=1}^m \int_0^t C_j^2 \tilde{u}(s) ds. \end{aligned}$$

Then  $\tilde{M}$  is an  $\mathbb{H}$ -valued continuous process.

*Proof.* Since  $\tilde{u} \in \mathcal{C}([0, T]; \mathbb{V})$  we just need to show that each of the remaining four terms on the RHS of (5.35) are  $\mathbb{H}$ -valued and well defined.

Using the Cauchy-Schwartz inequality repeatedly and by (5.29) we have the following inequalities

$$\tilde{\mathbb{E}} \int_0^T |A \tilde{u}(s)|_{\mathbb{H}} ds \leq T^{1/2} \left( \tilde{\mathbb{E}} \int_0^T |\tilde{u}(s)|_{\mathbb{D}(\mathbb{A})}^2 ds \right)^{1/2} < \infty.$$

Using (2.3), the Hölder inequality, (5.26) and the estimates (5.29) and (5.30) we obtain the following:

$$\begin{aligned}
\tilde{\mathbb{E}} \int_0^T |B(\tilde{u}(s))|_{\mathbf{H}} ds &\leq 2\tilde{\mathbb{E}} \int_0^T |\tilde{u}(s)|_{\mathbf{H}}^{1/2} |\nabla \tilde{u}(s)|_{L^2} |\tilde{u}(s)|_{\mathbf{D}(\mathbf{A})}^{1/2} ds \\
&\leq 2\tilde{\mathbb{E}} \left[ \left( \int_0^T \|\tilde{u}(s)\|_{\mathbf{V}}^{4/3} ds \right)^{3/4} \left( \int_0^T |\tilde{u}(s)|_{\mathbf{D}(\mathbf{A})}^2 ds \right)^{1/4} \right] \\
&\leq 2T^{3/4} \left( \tilde{\mathbb{E}} \sup_{s \in [0, T]} \|\tilde{u}(s)\|_{\mathbf{V}}^{4/3} \right)^{3/4} \left( \tilde{\mathbb{E}} \int_0^T |\tilde{u}(s)|_{\mathbf{D}(\mathbf{A})}^2 ds \right)^{1/4} < \infty.
\end{aligned}$$

Using the Hölder inequality, (5.26) and inequality (5.30) we have

$$\tilde{\mathbb{E}} \int_0^T |\nabla \tilde{u}(s)|_{L^2}^2 |\tilde{u}(s)|_{\mathbf{H}} ds \leq \tilde{\mathbb{E}} \int_0^T \|\tilde{u}(s)\|_{\mathbf{V}}^2 ds \leq \tilde{\mathbb{E}} \left( \sup_{s \in [0, T]} \|\tilde{u}(s)\|_{\mathbf{V}}^2 \right) T < \infty.$$

Now we are left to deal with the last term on the RHS. Using Assumption (A.1) and the estimate (5.29), we have the following inequalities for every  $j \in \{1, \dots, m\}$ ,

$$\tilde{\mathbb{E}} \int_0^T |C_j^2 \tilde{u}(s)|_{\mathbf{H}} ds \leq \frac{K_c T^{1/2}}{m} \left( \tilde{\mathbb{E}} \int_0^T |\tilde{u}(s)|_{\mathbf{D}(\mathbf{A})}^2 ds \right)^{1/2} < \infty.$$

This concludes the proof of the lemma.  $\square$

**Lemma 5.8.** *For all  $s, t \in [0, T]$  such that  $s \leq t$  then:*

- (a)  $\lim_{n \rightarrow \infty} \langle \tilde{u}_n(t), P_n \psi \rangle_{\mathbf{H}} = \langle \tilde{u}(t), \psi \rangle_{\mathbf{H}}, \quad \tilde{\mathbb{P}}\text{-a.s.} \quad \psi \in \mathbf{H},$
- (b)  $\lim_{n \rightarrow \infty} \int_s^t \langle A \tilde{u}_n(\sigma), P_n \psi \rangle_{\mathbf{H}} d\sigma = \int_s^t \langle A \tilde{u}(\sigma), \psi \rangle_{\mathbf{H}} d\sigma, \quad \tilde{\mathbb{P}}\text{-a.s.} \quad \psi \in \mathbf{H},$
- (c)  $\lim_{n \rightarrow \infty} \int_s^t \langle B(\tilde{u}_n(\sigma), \tilde{u}_n(\sigma)), P_n \psi \rangle_{\mathbf{H}} d\sigma = \int_s^t \langle B(\tilde{u}(\sigma), \tilde{u}(\sigma)), \psi \rangle_{\mathbf{H}} d\sigma, \quad \tilde{\mathbb{P}}\text{-a.s.} \quad \psi \in \mathbf{V},$
- (d)  $\lim_{n \rightarrow \infty} \int_s^t |\nabla \tilde{u}_n(\sigma)|_{L^2}^2 \langle \tilde{u}_n(\sigma), P_n \psi \rangle_{\mathbf{H}} d\sigma = \int_s^t |\nabla \tilde{u}(\sigma)|_{L^2}^2 \langle \tilde{u}(\sigma), \psi \rangle_{\mathbf{H}} d\sigma, \quad \tilde{\mathbb{P}}\text{-a.s.} \quad \psi \in \mathbf{H},$
- (e)  $\lim_{n \rightarrow \infty} \int_s^t \langle C_j^2 \tilde{u}_n(\sigma), P_n \psi \rangle_{\mathbf{H}} d\sigma = \int_s^t \langle C_j^2 \tilde{u}(\sigma), \psi \rangle_{\mathbf{H}} d\sigma, \quad \tilde{\mathbb{P}}\text{-a.s.} \quad \psi \in \mathbf{H}.$

*Proof.* Let us fix  $s, t \in [0, T]$ ,  $s \leq t$ . By (5.25) we know that

$$(5.36) \quad \tilde{u}_n \rightarrow \tilde{u} \text{ in } \mathcal{C}([0, T]; \mathbf{H}) \cap L_{\mathbf{w}}^2(0, T; \mathbf{D}(\mathbf{A})) \cap L^2(0, T; \mathbf{V}) \cap \mathcal{C}([0, T]; \mathbf{V}_{\mathbf{w}}), \quad \tilde{\mathbb{P}}\text{-a.s.}$$

Let  $\psi \in H$ . Since  $\tilde{u}_n \rightarrow \tilde{u}$  in  $\mathcal{C}([0, T]; H)$   $\tilde{\mathbb{P}}$ -a.s. and  $P_n \psi \rightarrow \psi$  in  $H$ , we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \langle \tilde{u}_n(t), P_n \psi \rangle_H - \langle \tilde{u}(t), \psi \rangle_H \\ &= \lim_{n \rightarrow \infty} \langle \tilde{u}_n(t) - \tilde{u}(t), P_n \psi \rangle_H + \lim_{n \rightarrow \infty} \langle \tilde{u}(t), P_n \psi - \psi \rangle_H = 0 \quad \tilde{\mathbb{P}}\text{-a.s.} \end{aligned}$$

Thus we infer that assertion (a) holds.

Let  $\psi \in H$ , then

$$\begin{aligned} & \int_s^t \langle A \tilde{u}_n(\sigma), P_n \psi \rangle_H d\sigma - \int_s^t \langle A \tilde{u}(\sigma), \psi \rangle_H d\sigma \\ &= \int_s^t \langle A \tilde{u}_n(\sigma) - A \tilde{u}(\sigma), \psi \rangle_H d\sigma + \int_s^t \langle A \tilde{u}_n(\sigma), P_n \psi - \psi \rangle_H d\sigma \\ &\leq \int_s^t \langle \tilde{u}_n(\sigma) - \tilde{u}(\sigma), A^{-1} \psi \rangle_{D(A)} d\sigma + \int_s^t |\tilde{u}_n(\sigma)|_{D(A)} |P_n \psi - \psi|_H d\sigma \\ &\leq \int_s^t \langle \tilde{u}_n(\sigma) - \tilde{u}(\sigma), A^{-1} \psi \rangle_{D(A)} d\sigma + |P_n \psi - \psi|_H |\tilde{u}_n|_{L^2(0, T; D(A))} T^{1/2}. \end{aligned}$$

By (5.36)  $\tilde{u}_n \rightarrow \tilde{u}$  weakly in  $L^2(0, T; D(A))$   $\tilde{\mathbb{P}}$ -a.s.  $\tilde{u}_n$  is a uniformly bounded sequence in  $L^2(0, T; D(A))$  and  $P_n \psi \rightarrow \psi$  in  $H$ . Hence we have,  $\tilde{\mathbb{P}}$ -a.s.,

$$\lim_{n \rightarrow \infty} \int_s^t \langle \tilde{u}_n(\sigma) - \tilde{u}(\sigma), A^{-1} \psi \rangle_{D(A)} d\sigma \rightarrow 0,$$

and

$$\lim_{n \rightarrow \infty} |P_n \psi - \psi|_H \rightarrow 0.$$

Thus, we have shown that assertion (b) is true.

We will now prove assertion (c). Let  $\psi \in V$ . Then we have the following estimates:

$$\begin{aligned} & \int_s^t \langle B(\tilde{u}_n(\sigma)), P_n \psi \rangle_H d\sigma - \int_s^t \langle B(\tilde{u}(\sigma)), \psi \rangle_H d\sigma \\ &= \int_s^t \langle B(\tilde{u}_n(\sigma)) - B(\tilde{u}(\sigma)), \psi \rangle_H d\sigma + \int_s^t \langle B(\tilde{u}_n(\sigma)), P_n \psi - \psi \rangle_H d\sigma \\ &= \int_s^t [b(\tilde{u}_n(\sigma), \tilde{u}_n(\sigma), \psi) - b(\tilde{u}(\sigma), \tilde{u}(\sigma), \psi)] d\sigma + \int_s^t \langle B(\tilde{u}_n(\sigma)), P_n \psi - \psi \rangle_H d\sigma. \end{aligned}$$

Using (2.2), we get

$$\begin{aligned}
& \int_s^t \langle B(\tilde{u}_n(\sigma)), P_n \psi \rangle_{\mathbf{H}} - \int_s^t \langle B(\tilde{u}(\sigma)), \psi \rangle d\sigma \\
&= \int_s^t b(\tilde{u}_n(\sigma) - \tilde{u}(\sigma), \tilde{u}_n(\sigma), \psi) d\sigma + \int_s^t b(\tilde{u}(\sigma), \tilde{u}_n(\sigma) - \tilde{u}(\sigma), \psi) d\sigma \\
&+ \int_s^t \langle B(\tilde{u}_n(\sigma)), P_n \psi - \psi \rangle d\sigma \\
&\leq \int_s^t \|\tilde{u}_n(\sigma) - \tilde{u}(\sigma)\|_{\mathbf{V}} \|\tilde{u}_n(\sigma)\|_{\mathbf{V}} \|\psi\|_{\mathbf{V}} d\sigma + \int_s^t \|\tilde{u}(\sigma)\|_{\mathbf{V}} \|\tilde{u}_n(\sigma) - \tilde{u}(\sigma)\|_{\mathbf{V}} \|\psi\|_{\mathbf{V}} d\sigma \\
&+ \int_s^t \|\tilde{u}_n(\sigma)\|_{\mathbf{V}}^2 \|P_n \psi - \psi\|_{\mathbf{V}} d\sigma.
\end{aligned}$$

Now since,  $\tilde{u}_n \rightarrow \tilde{u}$  in  $L^2(0, T; \mathbf{V})$ , in particular  $\tilde{u} \in L^2(0, T; \mathbf{V})$ , also the sequence  $(\tilde{u}_n)$  is uniformly bounded in  $L^2(0, T; \mathbf{V})$ . Thus using the Cauchy-Schwartz inequality and the convergence of  $P_n \psi \rightarrow \psi$  in  $\mathbf{V}$ , we have  $\mathbb{P}$ -a.s.,

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_s^t \langle B(\tilde{u}_n(\sigma)), P_n \psi \rangle_{\mathbf{H}} - \int_s^t \langle B(\tilde{u}(\sigma)), \psi \rangle d\sigma \\
&\leq \lim_{n \rightarrow \infty} \|\tilde{u}_n - \tilde{u}\|_{L^2(0, T; \mathbf{V})} \left[ \|\tilde{u}_n\|_{L^2(0, T; \mathbf{V})} \right. \\
&\quad \left. + \|\tilde{u}\|_{L^2(0, T; \mathbf{V})} \right] \|\psi\|_{\mathbf{V}} + \lim_{n \rightarrow \infty} \|\tilde{u}_n\|_{L^2(0, T; \mathbf{V})}^2 \|P_n \psi - \psi\|_{\mathbf{V}} \rightarrow 0.
\end{aligned}$$

Next we deal with (d). Let  $\psi \in \mathbf{H}$ , then

$$\begin{aligned}
& \int_s^t |\nabla \tilde{u}_n(\sigma)|_{L^2}^2 \langle \tilde{u}_n(\sigma), P_n \psi \rangle_{\mathbf{H}} d\sigma - \int_s^t |\nabla \tilde{u}(\sigma)|_{L^2}^2 \langle \tilde{u}(\sigma), \psi \rangle_{\mathbf{H}} d\sigma \\
&= \int_s^t [|\nabla \tilde{u}_n(\sigma)|_{L^2}^2 - |\nabla \tilde{u}(\sigma)|_{L^2}^2] \langle \tilde{u}(\sigma), \psi \rangle_{\mathbf{H}} d\sigma + \int_s^t |\nabla \tilde{u}_n(\sigma)|_{L^2}^2 \langle \tilde{u}_n(\sigma) - \tilde{u}(\sigma), \psi \rangle_{\mathbf{H}} d\sigma \\
&+ \int_s^t |\nabla \tilde{u}_n(\sigma)|_{L^2}^2 \langle \tilde{u}_n(\sigma), P_n \psi - \psi \rangle_{\mathbf{H}} d\sigma \\
&= \int_s^t [|\nabla \tilde{u}_n(\sigma)|_{L^2} - |\nabla \tilde{u}(\sigma)|_{L^2}] [|\nabla \tilde{u}_n(\sigma)|_{L^2} + |\nabla \tilde{u}(\sigma)|_{L^2}] \langle \tilde{u}(\sigma), \psi \rangle_{\mathbf{H}} d\sigma \\
&+ \int_s^t |\nabla \tilde{u}_n(\sigma)|_{L^2}^2 \langle \tilde{u}_n(\sigma) - \tilde{u}(\sigma), \psi \rangle_{\mathbf{H}} d\sigma + \int_s^t |\nabla \tilde{u}_n(\sigma)|_{L^2}^2 \langle \tilde{u}_n(\sigma), P_n \psi - \psi \rangle_{\mathbf{H}} d\sigma.
\end{aligned}$$

Thus by Cauchy-Schwartz inequality we get

$$\begin{aligned}
& \int_s^t |\nabla \tilde{u}_n(\sigma)|_{L^2}^2 \langle \tilde{u}_n(\sigma), P_n \psi \rangle_{\mathbf{H}} d\sigma - \int_s^t |\nabla \tilde{u}(\sigma)|_{L^2}^2 \langle \tilde{u}(\sigma), \psi \rangle_{\mathbf{H}} d\sigma \\
& \leq \int_s^t [\|\tilde{u}_n(\sigma) - \tilde{u}(\sigma)\|_{\mathbf{V}}] [\|\tilde{u}_n(\sigma)\|_{\mathbf{V}} + \|\tilde{u}(\sigma)\|_{\mathbf{V}}] |\tilde{u}(\sigma)|_{\mathbf{H}} |\psi|_{\mathbf{H}} d\sigma \\
& \quad + \int_s^t \|\tilde{u}_n(\sigma)\|_{\mathbf{V}}^2 |\tilde{u}_n(\sigma) - \tilde{u}(\sigma)|_{\mathbf{H}} |\psi|_{\mathbf{H}} d\sigma + \int_s^t \|\tilde{u}_n(\sigma)\|_{\mathbf{V}}^2 |\tilde{u}_n(\sigma)|_{\mathbf{H}} |P_n \psi - \psi|_{\mathbf{H}} d\sigma
\end{aligned}$$

By (5.36), since  $\tilde{u}_n \rightarrow \tilde{u}$  strongly in  $\mathcal{C}([0, T]; \mathbf{H}) \cap L^2(0, T; \mathbf{V})$ , in particular  $\tilde{u} \in L^2(0, T; \mathbf{V})$ , also the sequence  $(\tilde{u}_n)$  is uniformly bounded in  $L^2(0, T; \mathbf{V})$  and  $P_n \psi \rightarrow \psi$  in  $\mathbf{H}$ . Thus we have  $\tilde{\mathbb{P}}$ -a.s.

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_s^t |\nabla \tilde{u}_n(\sigma)|_{L^2}^2 \langle \tilde{u}_n(\sigma), P_n \psi \rangle_{\mathbf{H}} d\sigma - \int_s^t |\nabla \tilde{u}(\sigma)|_{L^2}^2 \langle \tilde{u}(\sigma), \psi \rangle_{\mathbf{H}} d\sigma \\
& \leq \lim_{n \rightarrow \infty} [\|\tilde{u}_n\|_{L^2(0, T; \mathbf{V})} + \|\tilde{u}\|_{L^2(0, T; \mathbf{V})}] \|\tilde{u}_n - \tilde{u}\|_{L^\infty(0, T; \mathbf{H})} \|\psi\|_{\mathbf{H}} \\
& \quad + \lim_{n \rightarrow \infty} \|\tilde{u}_n\|_{L^2(0, T; \mathbf{V})}^2 \|\tilde{u}_n - \tilde{u}\|_{L^\infty(0, T; \mathbf{H})} \|\psi\|_{\mathbf{H}} + \lim_{n \rightarrow \infty} \|\tilde{u}_n\|_{L^2(0, T; \mathbf{V})}^2 \|\tilde{u}_n\|_{L^\infty(0, T; \mathbf{H})} \|P_n \psi - \psi\|_{\mathbf{H}} \rightarrow 0.
\end{aligned}$$

Hence we infer that assertion (d) holds.

Now we are left to show that (e) holds. Let  $\psi \in \mathbf{H}$ , then

$$\begin{aligned}
& \int_s^t \langle C^2 \tilde{u}_n(\sigma), P_n \psi \rangle_{\mathbf{H}} d\sigma - \int_s^t \langle C^2 \tilde{u}(\sigma), \psi \rangle_{\mathbf{H}} d\sigma \\
& = \int_s^t \langle C^2 (\tilde{u}_n(\sigma) - \tilde{u}(\sigma)), \psi \rangle_{\mathbf{H}} d\sigma + \int_s^t \langle C^2 \tilde{u}_n(\sigma), P_n \psi - \psi \rangle_{\mathbf{H}} d\sigma \\
& \leq \int_s^t \langle C^2 A^{-1} A (\tilde{u}_n(\sigma) - \tilde{u}(\sigma)), \psi \rangle_{\mathbf{H}} d\sigma + K_c \int_s^t |\tilde{u}_n(\sigma)|_{\mathbf{D}(\mathbf{A})} |P_n \psi - \psi|_{\mathbf{H}} d\sigma,
\end{aligned}$$

where  $K_c$  is defined in Lemma 5.4.

Since  $(\tilde{u}_n)$  is a uniformly bounded sequence in  $L^2(0, T; \mathbf{D}(\mathbf{A}))$  and  $C^2 A^{-1}$  is a bounded operator thus by (5.36), we have  $\tilde{\mathbb{P}}$ -a.s.

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_s^t \langle C^2 \tilde{u}_n(\sigma), P_n \psi \rangle_{\mathbf{H}} d\sigma - \int_s^t \langle C^2 \tilde{u}(\sigma), \psi \rangle_{\mathbf{H}} d\sigma \\
& \leq \lim_{n \rightarrow \infty} \int_s^t \langle A (\tilde{u}_n(\sigma) - \tilde{u}(\sigma)), (C^2 A^{-1})^* \psi \rangle_{\mathbf{H}} d\sigma + \lim_{n \rightarrow \infty} K_c \|\tilde{u}\|_{L^2(0, T; \mathbf{D}(\mathbf{A}))} \|P_n \psi - \psi\|_{\mathbf{H}} T^{1/2} \\
& = \lim_{n \rightarrow \infty} \int_s^t \langle \tilde{u}_n(\sigma) - \tilde{u}(\sigma), A^{-1} (C^2 A^{-1})^* \psi \rangle_{\mathbf{D}(\mathbf{A})} d\sigma + \lim_{n \rightarrow \infty} K_c \|\tilde{u}\|_{L^2(0, T; \mathbf{D}(\mathbf{A}))} \|P_n \psi - \psi\|_{\mathbf{H}} T^{\frac{1}{2}} \rightarrow 0,
\end{aligned}$$



where to establish the convergence we have used that  $P_n\psi \rightarrow \psi$  in  $H$ .

This completes the proof of Lemma 5.8. □

Let  $h$  be a bounded continuous function on  $\mathcal{C}([0, T]; V_w)$ .

**Lemma 5.9.** *For all  $s, t \in [0, T]$ , such that  $s \leq t$  and all  $\psi \in V$ :*

$$(5.37) \quad \lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left[ \langle \tilde{M}_n(t) - \tilde{M}_n(s), \psi \rangle h(\tilde{u}_{n|[0,s]}) \right] = \tilde{\mathbb{E}} \left[ \langle \tilde{M}(t) - \tilde{M}(s), \psi \rangle h(\tilde{u}_{|[0,s]}) \right].$$

Here  $\langle \cdot, \cdot \rangle$  denotes the duality between  $V'$  and  $V$ .

*Proof.* Let us fix  $s, t \in [0, T]$ ,  $s \leq t$  and  $\psi \in V$ . By (5.31), we have

$$\begin{aligned} \langle \tilde{M}_n(t) - \tilde{M}_n(s), \psi \rangle &= \langle \tilde{u}_n(t), P_n\psi \rangle_H - \langle \tilde{u}_n(s), P_n\psi \rangle_H + \int_s^t \langle A\tilde{u}_n(\sigma), P_n\psi \rangle_H d\sigma \\ &+ \int_s^t \langle B(\tilde{u}_n(\sigma)), P_n\psi \rangle d\sigma - \int_s^t |\nabla \tilde{u}_n(\sigma)|_{L^2}^2 \langle \tilde{u}_n(\sigma), P_n\psi \rangle_H d\sigma \\ &- \frac{1}{2} \int_s^t \langle C^2 \tilde{u}_n(\sigma), P_n\psi \rangle_H d\sigma. \end{aligned}$$

By Lemma 5.8, we infer that

$$(5.38) \quad \lim_{n \rightarrow \infty} \langle \tilde{M}_n(t) - \tilde{M}_n(s), \psi \rangle = \langle \tilde{M}(t) - \tilde{M}(s), \psi \rangle, \quad \tilde{\mathbb{P}}\text{-a.s.}$$

In order to prove (5.37) we first observe that since  $\tilde{u}_n \rightarrow \tilde{u}$  in  $\mathcal{Z}_T$ , in particular in  $\mathcal{C}([0, T]; V_w)$  and  $h$  is a bounded continuous function on  $\mathcal{C}([0, T]; V_w)$ , we get

$$(5.39) \quad \lim_{n \rightarrow \infty} h(\tilde{u}_{n|[0,s]}) = h(\tilde{u}_{|[0,s]}) \quad \tilde{\mathbb{P}} - a.s.$$

and

$$(5.40) \quad \sup_{n \in \mathbb{N}} |h(\tilde{u}_{n|[0,s]})|_{L^\infty} < \infty.$$

Let us define a sequence of  $\mathbb{R}$ -valued random variables:

$$f_n(\omega) := \left[ \langle \tilde{M}_n(t, \omega), \psi \rangle - \langle \tilde{M}_n(s, \omega), \psi \rangle \right] h(\tilde{u}_{n|[0,s]}), \quad \omega \in \tilde{\Omega}.$$

We will prove that the functions  $\{f_n\}_{n \in \mathbb{N}}$  are uniformly integrable in order to apply the Vitali theorem later on. We claim that

$$(5.41) \quad \sup_{n \geq 1} \tilde{\mathbb{E}}[|f_n|^2] < \infty.$$

By the Cauchy-Schwartz inequality and the embedding  $V' \hookrightarrow H$ , for each  $n \in \mathbb{N}$  there exists a positive constant  $c$  such that

$$(5.42) \quad \tilde{\mathbb{E}}[|f_n|^2] \leq 2c|h|_{L^\infty}^2 |\psi|_V^2 \tilde{\mathbb{E}} \left[ |\tilde{M}_n(t)|_H^2 + |\tilde{M}_n(s)|_H^2 \right].$$

Since  $\tilde{M}_n$  is a continuous martingale with quadratic variation defined in (5.32), by the Burkholder-Davis-Gundy inequality we obtain

$$(5.43) \quad \tilde{\mathbb{E}} \left[ \sup_{t \in [0, T]} |\tilde{M}_n(t)|_H^2 \right] \leq c \tilde{\mathbb{E}} \left[ \sum_{j=1}^m \int_0^T |P_n C_j \tilde{u}_n(\sigma)|_H^2 d\sigma \right].$$

Since  $P_n: H \rightarrow H$  is a contraction then by Assumption (A.1) and (5.27) for  $p = 1$ , we have

$$(5.44) \quad \begin{aligned} \tilde{\mathbb{E}} \left[ \sum_{j=1}^m \int_0^T |P_n C_j \tilde{u}_n(\sigma)|_H^2 d\sigma \right] &\leq \tilde{\mathbb{E}} \left[ K_c \int_0^T \|\tilde{u}_n(\sigma)\|_V^2 d\sigma \right] \\ &\leq K_c \tilde{\mathbb{E}} \left[ \sup_{\sigma \in [0, T]} \|\tilde{u}_n(\sigma)\|_V^2 \right] T < \infty. \end{aligned}$$

Then by (5.42) and (5.44) we see that (5.41) holds. Since the sequence  $\{f_n\}_{n \in \mathbb{N}}$  is uniformly integrable and by (5.38) it is  $\tilde{\mathbb{P}}$ -a.s. point-wise convergent, then application of the Vitali Theorem completes the proof of the lemma.  $\square$

**Lemma 5.10.** *For all  $s, t \in [0, T]$  such that  $s \leq t$  and all  $\psi, \zeta \in V$ :*

$$\begin{aligned} \lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left[ \left( \langle \tilde{M}_n(t), \psi \rangle \langle \tilde{M}_n(t), \zeta \rangle - \langle \tilde{M}_n(s), \psi \rangle \langle \tilde{M}_n(s), \zeta \rangle \right) h(\tilde{u}_{n|[0, s]}) \right] \\ = \tilde{\mathbb{E}} \left[ \left( \langle \tilde{M}(t), \psi \rangle \langle \tilde{M}(t), \zeta \rangle - \langle \tilde{M}(s), \psi \rangle \langle \tilde{M}(s), \zeta \rangle \right) h(\tilde{u}_{|[0, s]}) \right], \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  denotes the dual pairing between  $V'$  and  $V$ .

*Proof.* Let us fix  $s, t \in [0, T]$  such that  $s \leq t$  and  $\psi, \zeta \in V$  and define the random variables  $f_n$  and  $f$  by

$$\begin{aligned} f_n(\omega) &:= \left( \langle \tilde{M}_n(t, \omega), \psi \rangle \langle \tilde{M}_n(t, \omega), \zeta \rangle - \langle \tilde{M}_n(s, \omega), \psi \rangle \langle \tilde{M}_n(s, \omega), \zeta \rangle \right) h(\tilde{u}_{n|[0,s]}(\omega)), \\ f(\omega) &:= \left( \langle \tilde{M}(t, \omega), \psi \rangle \langle \tilde{M}(t, \omega), \zeta \rangle - \langle \tilde{M}(s, \omega), \psi \rangle \langle \tilde{M}(s, \omega), \zeta \rangle \right) h(\tilde{u}_{|[0,s]}(\omega)), \quad \omega \in \tilde{\Omega}. \end{aligned}$$

By (5.38) and (5.39) we infer that  $\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega)$ , for  $\tilde{\mathbb{P}}$  almost all  $\omega \in \tilde{\Omega}$ .

We will prove that the functions  $\{f_n\}_{n \in \mathbb{N}}$  are uniformly integrable. We claim that for some  $r > 1$ ,

$$(5.45) \quad \sup_{n \geq 1} \tilde{\mathbb{E}} [|f_n|^r] < \infty.$$

For each  $n \in \mathbb{N}$ , as before we have

$$(5.46) \quad \tilde{\mathbb{E}} [|f_n|^r] \leq C \|h\|_{L^\infty}^r \|\psi\|_V^r \|\zeta\|_V^r \tilde{\mathbb{E}} \left[ |\tilde{M}_n(t)|^{2r} + |\tilde{M}_n(s)|^{2r} \right].$$

Since  $\tilde{M}_n$  is a continuous martingale with quadratic variation defined in (5.31), by the Burkholder-Davis-Gundy inequality we obtain

$$(5.47) \quad \tilde{\mathbb{E}} \left[ \sup_{t \in [0, T]} |\tilde{M}_n(t)|^{2r} \right] \leq c \tilde{\mathbb{E}} \left[ \sum_{j=1}^m \int_0^T |P_n C_j \tilde{u}_n(\sigma)|_{\mathbb{H}}^2 d\sigma \right]^r.$$

Since  $P_n : \mathbb{H} \rightarrow \mathbb{H}$  is a contraction, by Assumption (A.1) we have

$$\begin{aligned} \tilde{\mathbb{E}} \left[ \sum_{j=1}^m \int_0^T |P_n C_j \tilde{u}_n(\sigma)|_{\mathbb{H}}^2 d\sigma \right]^r &\leq \tilde{\mathbb{E}} \left[ K_c \int_0^T \|\tilde{u}_n(\sigma)\|_V^2 d\sigma \right]^r \\ (5.48) \quad &\leq K_c^r \tilde{\mathbb{E}} \left( \sup_{\sigma \in [0, T]} \|\tilde{u}_n(\sigma)\|_V^{2r} \right) T^r. \end{aligned}$$

Thus for  $r \in (1, 1 + \frac{1}{K_c})$ , by (5.46), (5.47), (5.48) and (5.27) we infer that condition (5.45) holds. By the Vitali theorem

$$(5.49) \quad \lim_{n \rightarrow \infty} \tilde{\mathbb{E}}[f_n] = \tilde{\mathbb{E}}[f].$$

The proof of the lemma is thus complete.  $\square$

**Lemma 5.11** (Convergence of quadratic variations). *For any  $s, t \in [0, T]$  and  $\psi, \zeta \in V$ , we have*

$$\begin{aligned} \lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left[ \left( \sum_{j=1}^m \int_s^t \langle (C_j \tilde{u}_n(\sigma))^* P_n \psi, (C_j \tilde{u}_n(\sigma))^* P_n \zeta \rangle_{\mathbb{R}} d\sigma \right) \cdot h(\tilde{u}_n|_{[0,s]}) \right] \\ = \tilde{\mathbb{E}} \left[ \left( \sum_{j=1}^m \int_s^t \langle (C_j \tilde{u}(\sigma))^* \psi, (C_j \tilde{u}(\sigma))^* \zeta \rangle_{\mathbb{R}} d\sigma \right) \cdot h(\tilde{u}|_{[0,s]}) \right]. \end{aligned}$$

*Proof.* Let us fix  $\psi, \zeta \in V$  and define a sequence of random variables by

$$f_n(\omega) := \left( \sum_{j=1}^m \int_s^t \langle (C_j \tilde{u}_n(\sigma, \omega))^* P_n \psi, (C_j \tilde{u}_n(\sigma, \omega))^* P_n \zeta \rangle_{\mathbb{R}} d\sigma \right) \cdot h(\tilde{u}_n|_{[0,s]}), \quad \omega \in \tilde{\Omega}.$$

We will prove that these random variables are uniformly integrable and convergent  $\tilde{\mathbb{P}}$ -a.s. to some random variable  $f$ . In order to do that we will show that for some  $r > 1$ ,

$$(5.50) \quad \sup_{n \geq 1} \tilde{\mathbb{E}} |f_n|^r < \infty.$$

Since  $P_n : H \rightarrow H$  is a contraction, by the Cauchy-Schwartz inequality, and Assumption (A.1) there exists a positive constant  $c$  such that

$$\begin{aligned} |(C_j \tilde{u}_n(\sigma, \omega))^* P_n \psi|_{\mathbb{R}} &\leq |(C_j \tilde{u}_n(\sigma, \omega))^*|_{L(H; \mathbb{R})} |P_n \psi|_H \leq |C_j \tilde{u}_n(\sigma, \omega)|_{L(\mathbb{R}; H)} |\psi|_H \\ &\leq K_c \|\tilde{u}_n(\sigma, \omega)\|_V |\psi|_H, \quad j \in \{1, \dots, m\}, \end{aligned}$$

where  $L(X, Y)$  denotes the operator norm of the linear operators from  $X$  to  $Y$ . Thus using the Hölder inequality, we obtain

$$\begin{aligned} \tilde{\mathbb{E}} |f_n|^r &= \tilde{\mathbb{E}} \left| \left( \sum_{j=1}^m \int_s^t \langle (C_j \tilde{u}_n(\sigma))^* P_n \psi, (C_j \tilde{u}_n(\sigma))^* P_n \zeta \rangle_{\mathbb{R}} d\sigma \right) \cdot h(\tilde{u}_n|_{[0,s]}) \right|^r \\ &\leq |h|_{L^\infty}^r \tilde{\mathbb{E}} \left( \sum_{j=1}^m \int_s^t |(C_j \tilde{u}_n(\sigma))^* P_n \psi|_{\mathbb{R}} \cdot |(C_j \tilde{u}_n(\sigma))^* P_n \zeta|_{\mathbb{R}} d\sigma \right)^r \\ &\leq K_c |h|_{L^\infty}^r |\psi|_H^r |\zeta|_H^r \tilde{\mathbb{E}} \left( \int_s^t \|\tilde{u}_n(\sigma)\|_V^2 d\sigma \right)^r \\ (5.51) \quad &\leq K_c |h|_{L^\infty}^r |\psi|_H^r |\zeta|_H^r \tilde{\mathbb{E}} \left( \sup_{\sigma \in [0, T]} \|\tilde{u}_n(\sigma)\|_V^{2r} \right) T^r. \end{aligned}$$

Therefore using (5.51) and (5.27) we infer that (5.50) holds for every  $r \in (1, 1 + \frac{1}{K_c})$ .  
Now for pointwise convergence we will show that for a fix  $\omega \in \tilde{\Omega}$ ,

$$(5.52) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \int_s^t \sum_{j=1}^m \langle (C_j \tilde{u}_n(\sigma, \omega))^* P_n \psi, (C_j \tilde{u}_n(\sigma, \omega))^* P_n \zeta \rangle_{\mathbb{R}} d\sigma \\ &= \int_s^t \sum_{j=1}^m \langle (C_j \tilde{u}(\sigma, \omega))^* \psi, (C_j \tilde{u}(\sigma, \omega))^* \zeta \rangle_{\mathbb{R}} d\sigma. \end{aligned}$$

Let us fix  $\omega \in \tilde{\Omega}$  such that

- (i)  $\tilde{u}_n(\cdot, \omega) \rightarrow \tilde{u}(\cdot, \omega)$  in  $L^2(0, T; V)$ ,
- (ii) and the sequence  $(\tilde{u}_n(\cdot, \omega))_{n \geq 1}$  is uniformly bounded in  $L^2(0, T; V)$ .

Note that to prove (5.52), it is sufficient to prove that

$$(5.53) \quad (C_j \tilde{u}_n(\sigma, \omega))^* P_n \psi \rightarrow (C_j \tilde{u}(\sigma, \omega))^* \psi \text{ in } L^2(s, t; \mathbb{R}),$$

for every  $j \in \{1, \dots, m\}$ . Using Cauchy-Schwartz we have

$$\begin{aligned} & \int_s^t |(C_j \tilde{u}_n(\sigma, \omega))^* P_n \psi - (C_j \tilde{u}(\sigma, \omega))^* \psi|_{\mathbb{R}}^2 d\sigma \\ & \leq \int_s^t \left( |(C_j \tilde{u}_n(\sigma, \omega))^* (P_n \psi - \psi)|_{\mathbb{R}} + |(C_j \tilde{u}_n(\sigma, \omega) - C_j \tilde{u}(\sigma, \omega))^* \psi|_{\mathbb{R}} \right)^2 d\sigma \\ & \leq 2 \int_s^t |C_j \tilde{u}_n(\sigma, \omega)|_{L(\mathbb{R}; H)}^2 |P_n \psi - \psi|_H^2 d\sigma + 2 \int_s^t |C_j \tilde{u}_n(\sigma, \omega) - C_j \tilde{u}(\sigma, \omega)|_{L(\mathbb{R}; H)}^2 |\psi|_H^2 d\sigma \\ & =: I_n^1(t) + I_n^2(t). \end{aligned}$$

We will deal with each of the terms individually. We start with  $I_n^1(t)$ . Since

$$\lim_{n \rightarrow \infty} |P_n \psi - \psi|_H = 0, \quad \psi \in V,$$

and by Assumption (A.1), (ii) there exists a positive constant  $K$  such that

$$\sup_{n \geq 1} \int_s^t |C \tilde{u}_n(\sigma, \omega)|_{L(\mathbb{R}; H)}^2 d\sigma \leq K_c \sup_{n \geq 1} \int_s^t \|\tilde{u}_n(\sigma, \omega)\|_V^2 d\sigma \leq K.$$

Thus we infer that

$$\lim_{n \rightarrow \infty} I_n^1(t) = 0.$$

Next we consider  $I_n^2(t)$ . Using Assumption (A.1) and (i) we can show that for every  $j \in \{1, \dots, m\}$ ,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_s^t |C_j \tilde{u}_n(\sigma, \omega) - C_j \tilde{u}(\sigma, \omega)|_{L(\mathbb{R}; \mathbb{H})}^2 |\psi|_{\mathbb{H}}^2 d\sigma \\ & \leq \lim_{n \rightarrow \infty} |\psi|_{\mathbb{H}}^2 K_1 \int_s^t \|\tilde{u}_n(\sigma, \omega) - \tilde{u}(\sigma, \omega)\|_{\mathbb{V}}^2 d\sigma = 0. \end{aligned}$$

Hence, we have proved (5.53), finishing the proof of lemma.  $\square$

By Lemma 5.9 we can pass to the limit in (5.33). By Lemmas 5.10 and 5.11 we can pass to the limit in (5.34) as well. After passing to the limits we infer that for all  $\psi, \zeta \in \mathbb{V}$ :

$$(5.54) \quad \tilde{\mathbb{E}} \left[ \langle \tilde{M}(t) - \tilde{M}(s), \psi \rangle h(\tilde{u}_{|[0,s]}) \right] = 0,$$

and

$$(5.55) \quad \begin{aligned} & \tilde{\mathbb{E}} \left[ \left( \langle \tilde{M}(t), \psi \rangle \langle \tilde{M}(t), \zeta \rangle - \langle \tilde{M}(s), \psi \rangle \langle \tilde{M}(s), \zeta \rangle \right. \right. \\ & \left. \left. - \sum_{j=1}^m \int_s^t \langle (C_j \tilde{u}(\sigma))^* \psi, (C_j \tilde{u}(\sigma))^* \zeta \rangle_{\mathbb{R}} d\sigma \right) \cdot h(\tilde{u}_{|[0,s]}) \right] = 0. \end{aligned}$$

**Theorem 3.4 proof continued.** Now we apply the idea analogous to that used by Da Prato and Zabczyk, see [8, Section 8.3]. By Lemma 5.7, (5.54) and (5.55) we infer that  $\tilde{M}(t), t \in [0, T]$  is a continuous square integrable martingale in  $\mathbb{H}$  with respect to the filtration  $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)$ , where  $\tilde{\mathcal{F}}_t = \sigma\{\tilde{u}(s), s \leq t\}$  with the quadratic variation

$$\langle \tilde{M} \rangle_t = \int_0^t \sum_{j=1}^m |C_j \tilde{u}(s)|_{\mathbb{H}}^2 ds.$$

Therefore by the Martingale Representation Theorem, there exist

- a stochastic basis  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_{t \geq 0}, \tilde{\mathbb{P}})$ ,
- a  $\mathbb{R}^m$ -valued  $\tilde{\mathbb{F}}$ -Wiener process  $\tilde{W}(t)$  defined on this basis,
- and a progressively measurable process  $\tilde{u}(t)$  such that for all  $t \in [0, T]$  and  $v \in \mathbb{V}$ :

$$\begin{aligned} & \langle \tilde{u}(t), v \rangle - \langle \tilde{u}(0), v \rangle + \int_0^t \langle A \tilde{u}(s), v \rangle ds + \int_0^t \langle B(\tilde{u}(s)), v \rangle ds \\ & = \int_0^t |\nabla \tilde{u}(s)|_{L_2}^2 \langle \tilde{u}(s), v \rangle ds + \frac{1}{2} \int_0^t \sum_{j=1}^m \langle C_j^2 \tilde{u}(s), v \rangle ds + \int_0^t \sum_{j=1}^m \langle C_j \tilde{u}(s), v \rangle d\tilde{W}(s). \end{aligned}$$

Thus the conditions from Definition 3.3 hold with  $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \geq 0}, \hat{\mathbb{P}}) = (\tilde{\tilde{\Omega}}, \tilde{\tilde{\mathcal{F}}}, \{\tilde{\tilde{\mathcal{F}}}_t\}_{t \geq 0}, \tilde{\tilde{\mathbb{P}}})$ ,  $\hat{W} = \tilde{\tilde{W}}$  and  $\hat{u} = \tilde{\tilde{u}}$ . Hence the proof of Theorem 3.4 is complete.

## 6. Pathwise uniqueness and strong solution

In this section we will show that the solutions of (3.1) are pathwise unique and that (3.1) has a strong solution. In the previous section we showed that the solution  $u$  of (3.1) lies in  $\mathcal{C}([0, T]; V_w) \cap L^2(0, T; D(A))$ . We start by proving Lemma 3.6, in particular showing  $u \in \mathcal{C}([0, T]; V) \cap L^2(0, T; D(A))$ .

*Proof of Lemma 3.6.*  $u$  is a martingale solution of (3.1) thus,  $u \in \mathcal{C}([0, T]; V_w) \cap L^2(0, T; D(A))$   $\hat{\mathbb{P}}$ -a.s. We start by showing that RHS of (3.5) makes sense. In order to do so we will show that each term on the RHS is well defined.

Firstly we consider the non-linear term arising from Navier-Stokes. Using (2.3), the Hölder inequality, (5.26) and (3.4), we have the following bounds :

$$\begin{aligned} \hat{\mathbb{E}} \int_0^T |B(u(s))|_{\mathbb{H}}^2 ds &\leq 2\hat{\mathbb{E}} \int_0^T |u(s)|_{\mathbb{H}} |\nabla u(s)|_{L^2}^2 |u(s)|_{D(A)} ds \\ &\leq 2T^{1/2} \left( \hat{\mathbb{E}} \sup_{s \in [0, T]} \|u(s)\|_V^4 \right)^{1/2} \left( \hat{\mathbb{E}} \int_0^T |u(s)|_{D(A)}^2 ds \right)^{1/2} < \infty. \end{aligned}$$

Using (5.26), the Hölder inequality, (5.25), (5.27) and (3.4) we have the following inequalities for the non-linear term generated from the projection of the Stokes operator,

$$\hat{\mathbb{E}} \int_0^T \left| |\nabla u(s)|_{L^2}^2 u(s) \right|_{\mathbb{H}}^2 ds = \hat{\mathbb{E}} \int_0^T |\nabla u(s)|_{L^2}^4 ds \leq T \left( \hat{\mathbb{E}} \sup_{s \in [0, T]} \|u(s)\|_V^4 \right) < \infty.$$

Next we deal with the correction term arising from the Stratonovich integral. Using Assumption (A.1) and (3.4), for every  $j \in \{1, \dots, m\}$  we have

$$\hat{\mathbb{E}} \int_0^T |C_j^2 u(s)|_{\mathbb{H}}^2 ds \leq K_c^2 \hat{\mathbb{E}} \int_0^T |u(s)|_{D(A)}^2 ds < \infty,$$

where  $K_c$  is defined in Lemma 5.4.

We are left to show that the Itô integral belongs to  $L^2(\Omega \times [0, T]; V)$ . Due to Itô symmetry it is enough to show that for every  $j \in \{1, \dots, m\}$

$$(6.1) \quad \hat{\mathbb{E}} \int_0^T \|C_j u(s)\|_V^2 ds < \infty.$$

Using Assumption (A.1) and (3.4), we have

$$\hat{\mathbb{E}} \int_0^T \|C_j u(s)\|_V^2 ds \leq C \hat{\mathbb{E}} \int_0^T |u(s)|_{D(A)}^2 ds < \infty.$$

Thus we have shown that each term in (3.5) is well defined. Now we will show that the equality holds.

Since  $u$  is a martingale solution of (3.1), for every  $v \in V$  and  $t \in [0, T]$  it satisfies the equality (3.2), i.e.  $\hat{\mathbb{P}}$ -a.s.

$$\begin{aligned} \langle u(t), v \rangle - \langle u_0, v \rangle + \int_0^t \langle Au(s), v \rangle ds + \int_0^t \langle B(u(s)), v \rangle ds \\ = \int_0^t |\nabla u(s)|_{L^2}^2 \langle u(s), v \rangle ds + \frac{1}{2} \int_0^t \sum_{j=1}^m \langle C_j^2 u(s), v \rangle ds + \int_0^t \sum_{j=1}^m \langle C_j u(s), v \rangle d\hat{W}_j(s). \end{aligned}$$

Note that the above equation holds true for every  $v \in \mathcal{V}$  (as defined in (2.1)) and hence (3.5) holds in the distribution sense. But since  $\mathcal{V}$  is dense in  $V$  (3.5) holds true almost everywhere.

We use [14, Lemma 4.1] to prove the first part of the lemma. We work with the  $D(A) \subset V \subset H$  space triple. Let us rewrite (3.5) in the following form

$$u(t) = u_0 + \int_0^t g(s) ds + N(t),$$

where  $g$  contains all the deterministic terms and  $N$  corresponds to the noise term. We have shown that  $g \in L^2(\Omega; L^2(0, T; H))$  and  $N \in L^2(\Omega; L^2(0, T; V))$ . Thus from [14, Lemma 4.1] we infer that  $u \in L^2(\Omega; C([0, T]; V))$ . This concludes the proof of lemma.  $\square$

In the following lemma we will prove that the solutions of (3.1) are pathwise unique. The proof uses the Schmalzfuss idea of application of the Itô formula for appropriate function (see [16]).

**Lemma 6.1.** *Assume that the assumptions (A.1) – (A.2) are satisfied. If  $u_1, u_2$  are two solutions of (3.1) defined on the same filtered probability space  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$  then  $\hat{\mathbb{P}}$ -a.s. for all  $t \in [0, T]$ ,  $u_1(t) = u_2(t)$ .*

*Proof.* Let us denote the difference of the two solutions by  $U := u_1 - u_2$ . Then  $U$  satisfies the following equation



$$(6.2) \quad dU(t) + [AU(t) + B(u_2(t)) - B(u_1(t))] dt = [|\nabla u_1(t)|_{L^2}^2 u_1(t) - |\nabla u_2(t)|_{L^2}^2 u_2(t)] dt + \sum_{j=1}^m C_j U(t) \circ dW_j(t), \quad t \in [0, T].$$

Let us define the stopping time

$$(6.3) \quad \tau_N := T \wedge \inf \{t \in [0, T] : \|u_1(t)\|_V^2 \vee \|u_2(t)\|_V^2 > N\}, \quad N \in \mathbb{N}.$$

Since  $\hat{\mathbb{E}} [\sup_{t \in [0, T]} \|u_i(t)\|_V^2] < \infty, \hat{\mathbb{P}}\text{-a.s. for } i = 1, 2, \lim_{N \rightarrow \infty} \tau_N = T.$

We apply the Itô formula to the function

$$F(t, x) = e^{-r(t)} |x|_{\mathbb{H}}^2, \quad t \in [0, T], x \in V$$

where  $r(t), t \in [0, T]$  is a real valued function which will be defined precisely later in the proof.

Since

$$\frac{\partial F}{\partial t} = -r'(t) e^{-r(t)} |x|_{\mathbb{H}}^2, \quad \frac{\partial F}{\partial x}(\cdot) = 2e^{-r(t)} \langle x, \cdot \rangle_{\mathbb{H}},$$

we obtain for all  $t \in [0, T]$

$$\begin{aligned} e^{-r(t \wedge \tau_N)} |U(t \wedge \tau_N)|_{\mathbb{H}}^2 &= \int_0^{t \wedge \tau_N} e^{-r(s)} \left( -r'(s) |U(s)|_{\mathbb{H}}^2 + 2 \langle -AU(s) + B(u_1(s)) - B(u_2(s)), U(s) \rangle_{\mathbb{H}} \right) ds \\ &\quad + \int_0^{t \wedge \tau_N} e^{-r(s)} \left( 2 \langle |\nabla u_1(s)|_{L^2}^2 u_1(s) - |\nabla u_2(s)|_{L^2}^2 u_2(s), U(s) \rangle_{\mathbb{H}} + \sum_{j=1}^m \langle C_j^2 U(s), U(s) \rangle_{\mathbb{H}} \right) ds \\ &\quad + \frac{1}{2} \int_0^{t \wedge \tau_N} \sum_{j=1}^m \text{Tr} \left[ C_j U(s) \frac{\partial^2 F}{\partial x^2} (C_j U(s))^* \right] ds + 2 \int_0^{t \wedge \tau_N} e^{-r(s)} \sum_{j=1}^m \langle C_j U(s), U(s) \rangle_{\mathbb{H}} dW(s). \end{aligned}$$

Thus using the Assumption (A.1), we obtain the following simplified expression

$$\begin{aligned} e^{-r(t \wedge \tau_N)} |U(t \wedge \tau_N)|_{\mathbb{H}}^2 &\leq \int_0^{t \wedge \tau_N} e^{-r(s)} \left( -r'(s) |U(s)|_{\mathbb{H}}^2 - 2 \|U(s)\|_V^2 - 2b(U(s), u_1(s), U(s)) \right) ds \\ &\quad + 2 \int_0^{t \wedge \tau_N} e^{-r(s)} \left( (|\nabla u_1(s)|_{L^2}^2 - |\nabla u_2(s)|_{L^2}^2) \langle u_1(s), U(s) \rangle_{\mathbb{H}} + |\nabla u_2(s)|_{L^2}^2 |U(s)|_{\mathbb{H}}^2 \right) ds \\ &\quad + \int_0^{t \wedge \tau_N} e^{-r(s)} \sum_{j=1}^m \left( \langle C_j^2 U(s), U(s) \rangle_{\mathbb{H}} + \frac{1}{2} \times 2 \langle C_j U(s), C_j U(s) \rangle_{\mathbb{H}} \right) ds. \end{aligned}$$

Using (2.2) and the Cauchy Schwartz inequality we get

$$\begin{aligned}
& e^{-r(t \wedge \tau_N)} |U(t \wedge \tau_N)|_{\mathbb{H}}^2 + 2 \int_0^{t \wedge \tau_N} e^{-r(s)} \|U(s)\|_{\mathbb{V}}^2 ds \\
& \leq \int_0^{t \wedge \tau_N} e^{-r(s)} \left( -r'(s) |U(s)|_{\mathbb{H}}^2 + 4 |U(s)|_{\mathbb{H}} \|U(s)\|_{\mathbb{V}} \|u_1(s)\|_{\mathbb{V}} \right) ds \\
& \quad + 2 \int_0^{t \wedge \tau_N} e^{-r(s)} \|U(s)\|_{\mathbb{V}} \left( |\nabla u_1(s)|_{L^2} + |\nabla u_2(s)|_{L^2} \right) |u_1(s)|_{\mathbb{H}} |U(s)|_{\mathbb{H}} ds \\
& \quad + 2 \int_0^{t \wedge \tau_N} e^{-r(s)} |\nabla u_2(s)|_{L^2}^2 |U(s)|_{\mathbb{H}}^2 ds.
\end{aligned}$$

Using Young's inequality we obtain

$$\begin{aligned}
(6.4) \quad & e^{-r(t \wedge \tau_N)} |U(t \wedge \tau_N)|_{\mathbb{H}}^2 + 2 \int_0^{t \wedge \tau_N} e^{-r(s)} \|U(s)\|_{\mathbb{V}}^2 ds \leq \int_0^{t \wedge \tau_N} e^{-r(s)} \left[ -r'(s) + 8 \|u_1(s)\|_{\mathbb{V}}^2 \right] |U(s)|_{\mathbb{H}}^2 ds \\
& \quad + 2 \int_0^{t \wedge \tau_N} e^{-r(s)} \left( |\nabla u_1(s)|_{L^2} + |\nabla u_2(s)|_{L^2} \right)^2 |u_1(s)|_{\mathbb{H}}^2 |U(s)|_{\mathbb{H}}^2 ds \\
& \quad + \int_0^{t \wedge \tau_N} e^{-r(s)} \|U(s)\|_{\mathbb{V}}^2 ds.
\end{aligned}$$

Now choosing

$$r(t) := \int_0^t \left[ 8 \|u_1(s)\|_{\mathbb{V}}^2 + 2 \left( |\nabla u_1(s)|_{L^2} + |\nabla u_2(s)|_{L^2} \right)^2 |u_1(s)|_{\mathbb{H}}^2 \right] ds,$$

inequality (6.4) reduces to

$$e^{-r(t \wedge \tau_N)} |U(t \wedge \tau_N)|_{\mathbb{H}}^2 + \int_0^{t \wedge \tau_N} e^{-r(s)} \|U(s)\|_{\mathbb{V}}^2 ds \leq 0.$$

In particular

$$(6.5) \quad \sup_{t \in [0, T]} \left[ e^{-r(t \wedge \tau_N)} |U(t \wedge \tau_N)|_{\mathbb{H}}^2 \right] = 0.$$

Note that since  $u_1$  and  $u_2$  are the martingale solutions of (3.1) satisfying the estimates (5.4) and (5.6) and because of the Lemma 5.1,  $r$  is well defined for all  $t \in [0, T]$ .

Since  $\hat{\mathbb{P}}$ -a.s.  $\lim_{N \rightarrow \infty} \tau_N = T$  and  $\hat{\mathbb{E}}[r(T)] < \infty$ , thus from (6.5) we infer that  $\hat{\mathbb{P}}$ -a.s. for all  $t \in [0, T]$ ,  $U(t) = 0$ . The proof of the lemma is thus complete.  $\square$

**Definition 6.2.** Let  $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, \mathbb{P}^i, W^i, u^i), i = 1, 2$  be the martingale solutions of (3.1) with  $u^i(0) = u_0, i = 1, 2$ . Then we say that the solutions are **unique in law** if

$$\text{Law}_{\mathbb{P}^1}(u^1) = \text{Law}_{\mathbb{P}^2}(u^2) \text{ on } \mathcal{C}([0, \infty); V_w) \cap L^2([0, \infty); D(A)),$$

where  $\text{Law}_{\mathbb{P}^i}(u^i), i = 1, 2$  are by definition probability measures on  $\mathcal{C}([0, \infty); V_w) \cap L^2([0, \infty); D(A))$ .

**Corollary 6.3.** Assume that assumptions (A.1) – (A.2) are satisfied. Then

- (1) There exists a pathwise unique strong solution of (3.1).
- (2) Moreover, if  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, u)$  is a strong solution of (3.1) then for  $\mathbb{P}$ -almost all  $\omega \in \Omega$  the trajectory  $u(\cdot, \omega)$  is equal almost everywhere to a continuous  $V$ -valued function defined on  $[0, T]$ .
- (3) The martingale solution of (3.1) is unique in law.

*Proof.* By Theorem 3.4 there exists a martingale solution and in the Lemma 6.1 we showed it is pathwise unique, thus assertion (1) follows from [13, Theorem 2]. Assertion (2) is a direct consequence of Lemma 3.6. Assertion (3) follows from [13, Theorems 2, 11].  $\square$

Using Theorem 3.4, Lemma 6.1 and Corollary 6.3 one can infer Theorem 3.8.

**Remark 6.4.** For any bounded Borel function  $\varphi \in \mathcal{B}_b(V)$  and  $t \geq 0$ , we define

$$(6.6) \quad (P_t \varphi)(u_0) = \mathbb{E}[\varphi(u(t, u_0))], \quad u_0 \in V.$$

Then one can show that this family of semigroups is sequentially Feller [5, Proposition 6.2]. In order to prove the existence of invariant measure following the idea from Maslowski-Seidler [10] one requires to obtain certain boundedness in probability which we haven't been able to establish so far. Thus proving the existence of invariant measure for Stochastic Constrained Navier-Stokes equations on  $\mathbb{T}^2$  is still open.

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